J. Math. Anal. Appl. 363 (2010) 174–181

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the conservation laws and invariant solutions of the mKdV equation

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ARTICLE INFO

Article history:
Received 6 June 2009
Available online 20 August 2009
Submitted by H. Frankowska

Dedicated to Prof. Nail H. Ibragimov on the occasion of his 70th birthday

Keywords:
Conservation laws
Symmetries

ABSTRACT

In this paper, we consider modified Korteweg–de Vries (mKdV) equation. By using the nonlocal conservation theorem method and the partial Lagrangian approach, conservation laws for the mKdV equation are presented. It is observed that only nonlocal conservation theorem method lead to the nontrivial and infinite conservation laws. In addition, invariant solution is obtained by utilizing the relationship between conservation laws and Lie-point symmetries of the equation.

1. Introduction

In all areas of physics, conservation laws are essential since they allow us to draw conclusions of a physical system under study in an efficient way. The famous laws of conservation of energy, linear momentum, and angular momentum are important tools for solving many problems arising in mathematical physics. Knowledge of conservation laws are important for numerical integration of partial differential equations (PDEs). Investigation of conservation laws of Korteweg–de Vries (KdV) equation became a starting point of discovery of new approaches to integration of PDEs (such as Miura transformations, Lax pairs, inverse scattering, bi-Hamiltonian structures, etc.). Existence of the sufficient number of conservation laws of (systems of) PDEs is a reliable indicator of their possible integrability. A variety of powerful methods, such as Noether's method [1,2], the direct construction formula method [3,4], the characteristic method [5], the variational approach (multiplier approach) [6], symmetry conditions method on the conserved quantities [7], partial Lagrangian method [8], Poisson brackets method [9], nonlocal conservation theorem method [10], have been used to investigate conservation laws of PDEs.

KdV equation

$$u_t = u_{xxx} + uu_x \tag{1}$$

is a mathematical model of waves on shallow water surfaces. Even though water waves are unstable in general [11], they do exhibit certain stability properties in approximate water wave models specific to certain regimes (such as KdV equation cf. the discussion in [12]). KdV equation is particularly famous as the prototypical example of an exactly solvable model, that is, a nonlinear partial differential equation whose solutions can be exactly and precisely specified. The solutions in turn include prototypical examples of solitons. KdV can be solved by means of the inverse scattering transform. The mathematical theory behind the KdV equation is rich and interesting, and, in the broad sense, is a topic of active mathematical research.

The modified KdV equation

$$u_t = u_{xxx} + u^2 u_x \tag{2}$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.08.030

differs from the original KdV equation in the last nonlinear term only. This, however, causes several substantial differences; the superficial similarity is rather inessential. On the other hand, these two equations are linked at a deeper level by the so called “Miura transformation”. Modified KdV equation arises in the process of understanding the role of nonlinear dispersion and in the formation of structures like liquid drops, and it exhibits compactons: solitons with compact support.

The main purpose of this paper is to find conservation laws of mKdV equation by using the nonlocal conservation theorem method and the partial Lagrangian approach. In addition, invariant solution is obtained by utilizing the relationship between conservation laws and Lie-point symmetries of the equation.

The outline of this paper is as follows: in Section 2, the fundamental relations are recalled. In Section 3, both approaches, i.e. nonlocal conservation theorem method and partial Lagrangian approach, to construct conservation laws are discussed. Section 4 describes double reductions from symmetries and conservation laws. In Section 5, conservation laws for mKdV equation are constructed with the aid of the both approaches. Section 6 is devoted to invariant solution of mKdV equation by utilizing the relationship between conservation laws and Lie-point symmetries. Finally, concluding remarks are given in Section 7.

2. Preliminaries

We first present notation to be used and recall basic definitions and theorems which can be found cited in the literature [13]. The summation convention is adopted in which there is summation over repeated upper and lower indices. Let x^i, $i = 1, 2, \dots, n$, be independent variables and u^α, $\alpha = 1, 2, \dots, N$, be N dependent variables. The derivatives of u^α with respect to x^i are $u^\alpha_i = D_i(u^\alpha)$, $u_{ij} = D_j D_i(u^\alpha)$, where

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \cdots, \quad i = 1, 2, \dots, n,$$

is the total derivative operator with respect to x^i. The collection of rth-order derivatives, $r \geqslant 1$, is denoted by $u_{(r)}$. As usual A is the vector space of differential functions. The basic operators defined in A are stated below.

The Euler–Lagrange operator is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geqslant 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \dots i_s}}, \quad \alpha = 1, 2, \dots, N,$$

and the Lie–Bäcklund operator is

$$\mathbf{X} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geqslant 1} \zeta^\alpha_{i_1 \dots i_s} \frac{\partial}{\partial u^\alpha_{i_1 \dots i_s}}, \tag{3}$$

where $\zeta^\alpha_{i_1 \dots i_s}$ are defined by

$$\begin{aligned} &\zeta^\alpha_i = D_i\big(\eta^\alpha\big) - u^\alpha_j D_i\big(\xi^j\big), \\ &\zeta^\alpha_{i_1 \dots i_s} = D_{i_s}\big(\zeta^\alpha_{i_1 \dots i_{s-1}}\big) - u_{j i_1 \dots i_{s-1}} D_{i_s}\big(\xi^j\big), \quad s > 1. \end{aligned} \tag{4}$$

The Noether operators associated with a Lie–Bäcklund operator $\mathbf{X}$ are

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u^\alpha_i} + \sum_{s \geqslant 1} D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \dots i_s}}, \quad i = 1, 2, \dots, n, \tag{5}$$

where Lie characteristic functions are

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j, \tag{6}$$

the Euler–Lagrange operator $\frac{\delta}{\delta u^\alpha_i}$ is

$$\frac{\delta}{\delta u^\alpha_i} = \frac{\partial}{\partial u^\alpha_i} + \sum_{s \geqslant 1} (-1)^s D_{j_1} \dots D_{j_s} \frac{\partial}{\partial u^\alpha_{i j_1 \dots j_s}}, \quad i = 1, 2, \dots, n, \ \alpha = 1, 2, \dots, N, \tag{7}$$

and similarly for the other Euler–Lagrange operators with respect to higher order derivatives.

Consider a kth-order system of differential equations of n independent and N dependent variables:

$$E_\alpha(x, u, u_{(1)} \dots, u_{(k)}) = 0, \quad \alpha = 1, 2, \dots, N. \tag{8}$$

A conserved vector of (8) is an n-tuple $T = (T^1, T^2, \dots, T^n)$, $T^i \in A$, $i = 1, 2, \dots, n$, such that:

$$D_i T^i = 0 \tag{9}$$

holds for all solutions of (8). Eq. (9) is called a local conservation law.

3. Methods to derive conservation laws

3.1. Nonlocal conservation theorem method

The system of adjoint equations to the system of kth-order differential equations (8) are defined by [14]

$$E_\alpha^*(x,u,v,\dots,u_{(k)},v_{(k)})=0,\quad \alpha=1,2,\dots,N, \tag{10}$$

where

$$E_\alpha^*(x,u,v,\dots,u_{(k)},v_{(k)})=\frac{\delta(v^\beta E_\beta)}{\delta u^\alpha},\quad \alpha=1,2,\dots,N,\qquad v=v(x) \tag{11}$$

and $v=(v^1,v^2,\dots,v^N)$ are new dependent variables.

Suppose system (8) admits the generator:

$$\mathbf{X}=\xi^i\frac{\partial}{\partial x^i}+\eta^\alpha\frac{\partial}{\partial u^\alpha}. \tag{12}$$

Then the adjoint system (11) admits the operator [10]:

$$\mathbf{Y}=\xi^i\frac{\partial}{\partial x^i}+\eta^\alpha\frac{\partial}{\partial u^\alpha}+\eta_*^\alpha\frac{\partial}{\partial v^\alpha},\quad \eta_*^\alpha=-\left(\lambda_\beta^\alpha+v^\alpha D_i\left(\xi^i\right)\right), \tag{13}$$

which is an extension of (12) to the variable v^α and λ_β^α are obtained from

$$\mathbf{X}(E_\alpha)=\lambda_\beta^\alpha E_\beta. \tag{14}$$

Theorem 3.1. *Every Lie point, Lie–Bäcklund and nonlocal symmetry of the system of kth-order differential equations* (8) *yields a conservation law for the system consisting of* (8) *and the adjoint equation* (10)*. The conserved vector components are*

$$T^i=\xi^i L+W^\alpha\frac{\delta L}{\delta u_i^\alpha}+\sum_{s\geqslant 1}D_{i_1}\dots D_{i_s}\frac{\partial}{\partial u_{i_1\dots i_s}^\alpha} \tag{15}$$

with Lagrangian given by

$$L=v^\alpha E_\alpha(x,u,\dots,u_{(k)}) \tag{16}$$

and ξ^i, η^α are the coefficient functions of the generator (12)*. The conserved vectors obtained from* (15) *involves the arbitrary solutions v of the adjoint equation* (10) *and hence one obtains an infinite number of conservation laws for* (8) *by specifying v* [10].

3.2. Partial Noether approach

If the standard Lagrangian does not exist or is difficult to find, then we write its partial Lagrangian and derive the conservation laws by the partial Noether approach introduced by Kara and Mahomed [8].

Suppose that the kth-order differential system (8) can be written as

$$E_\alpha=E_\alpha^0+E_\alpha^1=0. \tag{17}$$

A function $L=L(x,u,u_{(1)},u_{(2)},\dots,u_{(l)})$, $l\leqslant k$ is called a partial Lagrangian of system (8) if system (8) can be expressed as $\frac{\delta L}{\delta u^\alpha}=f_\alpha^\beta E_\beta^1$ provided $E_\beta^1\neq 0$ for some β.

The operator X defined in (3) satisfying

$$\mathbf{X}(L)+LD_i\left(\xi^i\right)=D_i\left(B^i\right)+\left(\eta^\alpha-\xi^j u_j^\alpha\right)\frac{\delta L}{\delta u^\alpha},\quad i=1,2,\dots,N, \tag{18}$$

is a partial Noether operator corresponding to the partial Lagrangian L.

The conserved vector of system (8) associated with a partial Noether operator $\mathbf{X}$ corresponding to the partial Lagrangian L is determined from

$$T^i=B^i-N^iL=B^i-\xi^i L-W^\alpha\frac{\delta L}{\delta u_i^\alpha}-\sum_{s\geqslant 1}D_{i_1\dots i_s}\left(W^\alpha\right)\frac{\partial}{\partial u_{i_1\dots i_s}^\alpha}. \tag{19}$$

Here also W^α are the characteristics of the conservation law.

4. Double reduction method of PDEs from the association of symmetries with conservation laws

If $\mathbf{X}$ and T satisfy

$$\mathbf{X}(T^i) + T^i D_j(\xi^j) - T^j D_j(\xi^i) = 0, \quad i = 1, 2, \tag{20}$$

then $\mathbf{X}$ is associated with T.

We define a nonlocal variable w by $T^t = w_x$, $T^x = -w_t$. In the similarity variables $T^r = w_s$, $T^s = -w_r$, so that the conservation law is rewritten as

$$D_r T^r + D_s T^s = 0$$

with

$$T^s = \frac{T^t D_t(s) + T^x D_x(s)}{D_t(r) D_x(s) - D_x(r) D_t(s)} \tag{21}$$

and

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_t(r) D_x(s) - D_x(r) D_t(s)}. \tag{22}$$

The components T^x, T^t depend upon $(x, t, u, u_{(1)}, u_{(2)}, \ldots, u_{(q-1)})$ which means that T^s, T^r depend upon $(s, r, \theta, \theta_r, \theta_{rr}, \ldots, \theta_{r^{q-1}})$ for solutions invariant under $\mathbf{X}$. Therefore $D_r T^r + D_s T^s = 0$ become $\frac{\partial T^s}{\partial s} + D_r T^r = 0$ or

$$T^r = \int \frac{\partial T^s}{\partial s}\, dr + f(s).$$

For T associated with $\mathbf{X}$ we have $\mathbf{X}T^r = 0$ and $\mathbf{X}T^s = 0$. Thus T^r and T^s are invariant under $\mathbf{X}$. This means

$$\frac{\partial}{\partial s} T^r = 0 \quad \text{and} \quad \frac{\partial}{\partial s} T^s = 0.$$

The conservation law in canonical coordinates becomes

$$D_r T^r = 0.$$

A PDE $E = 0$ of order q with two independent variables, which admits a symmetry $\mathbf{X}$ that is associated with a conserved vector T, is reduced to an ODE of order $q - 1$, namely $T^r = k$, where T^r is given by (22) for solutions invariant under $\mathbf{X}$ [15].

5. Conservation laws of the mKdV equation

Consider now the nonlocal conservation theorem method given by Ibragimov [10]. MKdV equation admits the following Lie-point symmetry generators:

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \qquad \mathbf{X}_2 = \frac{\partial}{\partial t}, \qquad \mathbf{X}_3 = -x\frac{\partial}{\partial x} - 3t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}. \tag{23}$$

Eq. (2) has not the usual Lagrangian. The adjoint equation for (2) is

$$E^*(t, x, u, v, \ldots, v_{xxx}) = \frac{\delta}{\delta u}\big[(u_t - u_{xxx} - u^2 u_x)v\big] = 0, \quad v = v(t, x) \tag{24}$$

which yields

$$E^* = v_t - u^2 v_x - v_{xxx} = 0 \tag{25}$$

where v is the adjoint variable. If one substitutes u instead of v in Eq. (25), Eq. (2) is obtained. Consequently, Eq. (2) is self-adjoint. Now, consider Eq. (2) and the adjoint equation (25) as a system. The Lagrangian for the system is, from (16),

$$L = (u_t - u_{xxx} - u^2 u_x)v \tag{26}$$

that is,

$$\frac{\delta L}{\delta v} = u_t - u_{xxx} - u^2 u_x, \qquad \frac{\delta L}{\delta u} = v_t - v^2 v_x - v_{xxx}. \tag{27}$$

In addition, Eq. (25) admits Lie-point symmetry generators (23). Let us illustrate this fact for $\mathbf{X}_3$.

If one verify symmetry invariance condition for E^* on $\mathbf{X}_3$, it is seen that,

$$\widetilde{\mathbf{X}^3}(E^*)\big|_{E^*=0} = 3(-v_t + u^2 v_x + v_{xxx})$$

with $\widetilde{\mathbf{X}^3} = -x\frac{\partial}{\partial x} - 3t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + 3v_t\frac{\partial}{\partial v_t} + v_x\frac{\partial}{\partial v_x} + 2v_{xx}\frac{\partial}{\partial v_{xx}} + 3v_{xxx}\frac{\partial}{\partial v_{xxx}}$, where $\widetilde{\mathbf{X}^3}$ is the prolonged vector of $\mathbf{X}_3$.

Eq. (2) also satisfies invariance test

$$\mathbf{X}(L) + LD_i(\xi^i) = 0 \tag{28}$$

for each Lie-point symmetry in (23). Let us try to show for $\mathbf{X}_1 = \frac{\partial}{\partial x}$. Here $\xi = 1$, $\tau = 0$ and therefore $D_t(\tau) + D_x(\xi)$ is zero. $\mathbf{X}(L)$ is also zero for the Lagrangian (26). If we substitute these values into Eq. (28), invariance condition is clearly satisfied.

The conserved vectors of the system Eqs. (2) and (25), associated with a symmetry, can be obtained from (15) as follows:

$$T^1 = \tau L + W\frac{\partial L}{\partial u_t}, \tag{29}$$

$$T^2 = \xi L + W\left[\frac{\partial L}{\partial u_x} + D_x^2\left(\frac{\partial L}{\partial u_{xxx}}\right)\right] - D_x(W)\left[D_x\left(\frac{\partial L}{\partial u_{xxx}}\right)\right] + D_x^2(W)\frac{\partial L}{\partial u_{xxx}} \tag{30}$$

where

$$W = \eta - \tau u_t - \xi u_x. \tag{31}$$

Let us construct conservation laws corresponding to symmetries.

Case 1. First, we consider $\mathbf{X}_1 = \frac{\partial}{\partial x}$. Eq. (13) reveals that $\mathbf{Y}_1$ coincides with $\mathbf{X}_1$. The infinitesimals are $\xi = 1$, $\tau = 0$, $\eta = 0$ and the Lie characteristic function is $W = -u_x$ and conserved vectors from (29) and (30) are

$$T^1 = -vu_x,$$

$$T^2 = vu_t + u_x v_{xx} - v_x u_{xx}.$$

If we take $v = u$ in the above, because of self-adjointness of Eq. (2), conserved vectors are as follows:

$$T^1 = -uu_x = -D_x\left(\frac{u^2}{2}\right),$$

$$T^2 = uu_t = D_t\left(\frac{u^2}{2}\right). \tag{32}$$

It is obvious that, if we use conserved quantities (32) in (9), trivial conservation laws are obtained.

Case 2. Second, we consider $\mathbf{X}_2 = \frac{\partial}{\partial t}$. The infinitesimals are $\xi = 0$, $\tau = 1$, $\eta = 0$ and the Lie characteristic function is $W = -u_t$ and conserved vectors from (29) and (30) are

$$T^1 = -uu_{xxx},$$

$$T^2 = u_t u_{xx} - u_x u_{xt} + uu_{txx}. \tag{33}$$

We note that $-uu_{xxx} = D_x(\frac{1}{2}u_x^2 - uu_{xx})$ and $u_t u_{xx} - u_x u_{xt} + uu_{txx} = D_t(uu_{xx} - \frac{1}{2}u_x^2)$. Then it is readily seen that if we use conserved quantities (33) in (9), again trivial conservation laws are obtained.

Case 3. Third, we consider $\mathbf{X}_3 = -x\frac{\partial}{\partial x} - 3t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$. The infinitesimals are $\xi = -x$, $\tau = -3t$, $\eta = u$ and the Lie characteristic function is $W = u + xu_x + 3tu_t$ and nontrivial conserved vectors from (29) and (30) are

$$T^1 = 3tuu_{xxx} + 3tu^3u_x + u^2 + xuu_x,$$

$$T^2 = -4uu_{xx} - xuu_t - 3tu^3u_t - 3tu_t u_{xx} + 2u_x^2 + 3tu_x u_{xt} - 3tuu_{xxt} - u^4. \tag{34}$$

We point out that, Ibragimov's approach is not, only verified for Lie-point type symmetry but also is valid for Lie–Bäcklund type and nonlocal type symmetries.

Let us consider the following Lie–Bäcklund operators are [16]

$$\mathbf{X} = f^{(i)}(t, x, u, u_1, u_2, \ldots)\frac{\partial}{\partial u} + \cdots, \quad u_i = D_x^i u, \tag{35}$$

where $R = D_x^2 + \frac{2}{3}u^2 + \frac{2}{3}u_x D_x^{-1}.\ u$ is the recursion operator. (The last summand is the operator that takes a differential function, multiplies it by u, then takes D_x^{-1} (if possible) and finally multiplies the result by $\frac{2}{3}u_x$.) For $i = 5$

$$f^{(5)} = \left[u_5 + \frac{5}{3}u^2u_3 + \frac{20}{3}uu_1u_2 + \frac{5}{3}u_1^3 + \frac{5}{6}u^4u_1\right]$$

and conserved density is

$$T^1 = uf^{(5)}. \tag{36}$$

Notice that, we can obtain infinite number conservation laws which base on the Lie–Bäcklund symmetry (35). In the Ibragimov's method every symmetry of the given equation provides a conservation law, it might be trivial (zero components), but always a conservation law exists.

Consider now the partial Lagrangian approach given by Kara and Mahomed [8]. The mKdV equation has no partial Lagrangian unless one (as is done for standard Lagrangians) lets $u = v_x$ in which case

$$v_{tx} = v_{xxxx} + v_x^2 v_{xx}. \tag{37}$$

A partial Lagrangian for the latter equation is $L = \frac{v_t v_x}{2} + \frac{v_{xx}^2}{2}$ and the Euler–Lagrange-type equation is $\frac{\delta L}{\delta v} + v_{xt} - v_{xxxx} = 0$ so that Eq. (37) can be written as $\frac{\delta L}{\delta v} = -v_x^2 v_{xx}$.

The partial Noether symmetry determining equation is, by (18),

$$\mathbf{X}^{[2]} L + (D_t \tau + D_x \xi) L = D_t B^1 + D_x B^2 + (\eta - \tau v_t - \xi v_x) \frac{\delta L}{\delta v} \tag{38}$$

Eq. (38) for $L = \frac{v_t v_x}{2} + \frac{v_{xx}^2}{2}$ gives:

$$\begin{aligned}
&\frac{v_x}{2}\big[\eta_t + v_t \eta_v - v_x \xi_t - v_x v_t \xi_v - v_t \tau_t - v_t^2 \tau_v\big] \\
&\quad + \frac{v_t}{2}\big[\eta_x + v_x \eta_v - v_x \xi_x - v_x v_t \tau_v - v_t \tau_x - v_x^2 \xi_v\big] \\
&\quad + v_{xx}\big(\eta_{xx} + 2 v_x \eta_{xv} + v_{xx} \eta_v + v_x^2 \eta_{vv} - 2 v_{xx} \xi_x - v_x \xi_{xx} \\
&\quad - 2 v_x^2 \xi_{xv} - 3 v_x v_{xx} \xi_v - v_x^3 \xi_{vv} - 2 v_{xt} \tau_x \\
&\quad - v_t \tau_{xx} - 2 v_x v_t \tau_{xv} - v_t v_{xx} \tau_v - 2 v_x v_{xt} \tau_v - v_x^2 v_t \tau_{vv}\big) \\
&\quad + \left(\frac{v_t v_x}{2} + \frac{v_{xx}^2}{2}\right)[\xi_x + v_x \xi_v + \tau_t + v_t \tau_v] \\
&= -\eta v_x^2 v_{xx} + v_x^3 v_{xx} \xi + v_x^2 v_t v_{xx} \tau + B_t^1 + v_t B_v^1 + B_x^2 + v_x B_v^2
\end{aligned} \tag{39}$$

where $B^1 = B^1(t, x, v)$ and $B^2 = B^2(t, x, v)$ are the gauge terms. Separating Eq. (39) with respect to derivatives of v yield the following overdetermined linear system

$$\begin{aligned}
&v_x : \frac{\eta_t}{2} - B_v^2 = 0, \\
&v_x v_t : \eta_v + \frac{\xi_x}{2} = 0, \\
&v_x^2 : \xi_t = 0, \\
&v_x^2 v_t : \xi_v = 0, \\
&v_x v_t^2 : \tau_v = 0, \\
&v_t : \frac{\eta_x}{2} - B_v^1 = 0, \\
&v_t^2 : \tau_x = 0, \\
&v_{xx} : \eta_{xx} = 0, \\
&v_x v_{xx} : 2\eta_{xv} - \xi_{xx} = 0, \\
&v_{xx}^2 : \eta_v - 3\frac{\xi_x}{2} + \frac{\tau_t}{2} = 0, \\
&v_x^2 v_{xx} : \eta_{vv} - 2\xi_{xv} + \eta = 0, \\
&v_x^3 v_{xx} : \xi_{vv} + \xi = 0, \\
&v_{xt} v_{xx} : \tau_x = 0, \\
&v_t v_{xx} : \tau_{xx} = 0, \\
&v_x v_t v_{xx} : \tau_{xv} = 0, \\
&v_x^2 v_t v_{xx} : \tau_{vv} + \tau = 0, \\
&1 : B_t^1 + B_x^2 = 0.
\end{aligned} \tag{40}$$

The calculations reveal that $\xi(x,t,v) = \tau(x,t,v) = \eta(x,t,v) = 0$. If one choose partial Lagrangian as $L = \frac{v_{xx}^2}{2} - \frac{1}{12} v_x^4$, and $\frac{\delta L}{\delta v} = v_{tx}$ the calculations again show that $\xi(x,t,v) = \tau(x,t,v) = \eta(x,t,v) = 0$. The partial Lagrangian approach is an extension of the Noether approach and as seen in this situation it may lead to no infinitesimals as in the Noether case. So it is dependent very much on the choice of a partial Lagrangian.

6. Invariant solution of the mKdV equation

Eq. (2), admits the symmetry generators $\mathbf{X}_1 = \frac{\partial}{\partial x}$, $\mathbf{X}_2 = \frac{\partial}{\partial t}$ associated with the conservation law $D_t(u) + D_x(-u_{xx} - \frac{u^3}{3}) = 0$ in the sense of (20).

Let $\mathbf{X} = \mathbf{X}_1 + c\mathbf{X}_2$. Canonical coordinates of X (such that $\mathbf{X} = \frac{\partial}{\partial s}$) are

$$s = x, \qquad r = cx - t, \qquad u. \tag{41}$$

The above mentioned conservation law is written as $D_s T + D_r T^r = 0$ with

$$T^r = \frac{u D_t(x) + (-u_{xx} - \frac{u^3}{3}) D_x(x)}{D_t(cx - t) D_x(x) - D_x(cx - t) D_t(x)} = u_{xx} + \frac{u^3}{3} = c^2 u_{rr} + \frac{u^3}{3}. \tag{42}$$

Since $T = (T^r, T^s)$ is associated with $\mathbf{X}$,

$$T^r = k_1 \Rightarrow c^2 u_{rr} + \frac{u^3}{3} = k_1. \tag{43}$$

The reduced equation is

$$c^2 u_{rr} + \frac{u^3}{3} = k_1. \tag{44}$$

We write u_{rr} as $\frac{du_r}{du} u_r$ so that the above equation becomes

$$u_r du_r = \frac{k_1 - \frac{u^3}{3}}{c^2} du \tag{45}$$

and therefore

$$u_r = \left(\frac{2k_1}{c^2} u - \frac{u^4}{6c^2} + c_1 \right)^{\frac{1}{2}}. \tag{46}$$

A second integration to r leads to

$$\int \left(\frac{2k_1}{c^2} u - \frac{u^4}{6c^2} + c_1 \right)^{-\frac{1}{2}} du = r + c_2 \tag{47}$$

is a 4 parameter family of solutions of Eq. (2) invariant under $\mathbf{X} = \mathbf{X}_1 + c\mathbf{X}_2$.

7. Conclusions

In this paper, we considered modified Korteweg–de Vries (mKdV) equation. By using the nonlocal conservation theorem method and the partial Lagrangian approach, conservation laws for the mKdV equation are discussed. It is observed that only nonlocal conservation theorem method lead to the nontrivial conservation law (34) for Lie point symmetries and infinite conservation densities (one of them is Eq. (36)) for Lie–Bäcklund symmetry. In addition, invariant solution (47) is obtained by utilizing the relationship between conservation laws and Lie-point symmetries of the equation.

Acknowledgments

The author would like to express sincerely thanks to the referees careful readings and useful suggestions on the additional references.

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