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Fully invariant-extending modular lattices, and applications (I)

Toma Albu [a,*], Yeliz Kara [b], Adnan Tercan [c]

[a] *Simion Stoilow Institute of Mathematics of the Romanian Academy, P.O. Box 1 - 764, RO - 010145 Bucharest 1, Romania*
[b] *Uludağ University, Department of Mathematics, 16059 Bursa, Turkey*
[c] *Hacettepe University, Department of Mathematics, Beytepe Campus, 06532 Ankara, Turkey*

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ABSTRACT

Based on the concept of a linear morphism of lattices, recently introduced in the literature, we introduce and investigate in this paper the latticial counterpart of the notion of a fully invariant-extending module.

* Corresponding author.
E-mail addresses: Toma.Albu@imar.ro (T. Albu), yelizkara@uludag.edu.tr (Y. Kara), tercan@hacettepe.edu.tr (A. Tercan).

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Introduction

In recent years the theory of extending modules has become an important part of Module Theory (see [17]). Recall that a module M is called an *extending* (or a *CS*) module if every submodule of M is essential in a direct summand of M. Unfortunately, the class of these modules is not closed under direct sums. Therefore, in order to avoid this unpleasant situation, it is natural to relax the definition above of the extending property by replacing the set $\mathcal{L}(M)$ of *all* submodule of M with a suitable subset $\mathcal{S}(M)$ of it. More precisely, let us call a module M as being $\mathcal{S}(M)$-*extending* if every member of $\mathcal{S}(M)$ is essential in a direct summand of M, and find relevant subsets of $\mathcal{L}(M)$ having the property that any direct sum of $\mathcal{S}(M)$-*extending* modules is $\mathcal{S}(M)$-*extending* too.

Such a subset $\mathcal{S}(M)$ in this context has been considered for the first time in [12] as the set $\mathcal{FI}(M)$ of all fully invariant submodules of M. Recall that a submodule N of M is said to be *fully invariant* if $f(N) \subseteq N$ for every $f \in \mathrm{End}_R(M)$. The set of fully invariant submodules includes many of the most significant submodules of a module (e.g., the Jacobson radical, the socle, the singular submodule of M, etc.). These $\mathcal{FI}(M)$-extending modules have been called *FI-extending* modules in [12].

The extending modules are exactly those having the condition (C_1). Two more conditions, namely (C_2), and (C_3), have been introduced in the literature in the 1960's, first for rings and then for modules, and, in combination with the condition (C_1), the important classes of *continuous* and *quasi-continuous* have been defined and investigated (see the seminal monograph [21] devoted to them). In the 1990's, the conditions (C_{11}) and (C_{12}) for modules have been introduced and investigated in [24] as generalizations of the condition (C_1); for these and other generalizations of extending modules, see the monograph [26].

Very recently, the known conditions (C_i), $i = 1, 2, 3, 11, 12$, on modules have been defined and investigated in [10] in a latticial frame, in order to apply them to Grothendieck categories and module categories equipped with hereditary torsion theories. Notice that the well-known *Osofsky–Smith Theorem* [22] of Module Theory has a latticial counterpart, with nice categorical and torsion-theoretical applications (see [2] and [3]).

In the same trend, we are interested in this paper to introduce and investigate the latticial counterpart of the concept of a FI-extending module. Our main tool in doing so is the very useful concept of *linear morphism of lattices* introduced in [5].

This paper, as well as our other articles mentioned above, again illustrate a general strategy which consists on putting a module-theoretical definition/result in a latticial frame, in order to translate that definition/result to Grothendieck categories and to module categories equipped with a hereditary torsion theory.

In Section 0 we present some general terminology and notation about lattices, especially from [4], as well as the definition and basic properties of linear morphisms of lattices introduced in [5]; these morphisms play a major role in the whole paper.

In Section 1 we introduce the concepts of a fully invariant element of a lattice and of a fully invariant-extending, shortly, FI-extending, lattice and present some of their basic properties.

Section 2 is devoted to the main results of this paper. Specifically, we prove that the latticial FI-extending condition is inherited by direct joins, and discuss whether this condition is inherited under complement intervals.

Applications of our latticial results to Grothendieck categories and module categories equipped with a hereditary torsion theory will be given in a subsequent paper.

0. Preliminaries

All lattices considered in this paper are assumed to be *bounded,* i.e., to have a least element denoted by 0 and a last element denoted by 1, and L will always denote such a lattice. If the lattices L and L' are isomorphic, we denote this by $L \simeq L'$.

We denote by $\mathcal{L}$ (respectively, $\mathcal{M}$) the class of all bounded (respectively, bounded modular) lattices.

For a lattice L and elements $a \leqslant b$ in L we write

$$b/a := [a, b] = \{\ x \in L \mid a \leqslant x \leqslant b\ \}.$$

A *subfactor* of L is any interval b/a of L with $a \leqslant b$. An *initial interval* of b/a is any interval c/a for some $c \in b/a$.

For basic notation and terminology on lattices the reader is referred to [4], [15], [16], [19], and/or [25], but especially to [4]. In particular, for any $L \in \mathcal{L}$, one denote

$E(L)$:= the set of all *essential* elements of L (E for "*Essential*"),
$C(L)$:= the set of all *closed* elements of L (C for "*Closed*"),
$P(L)$:= the set of all *pseudo-complement* elements of L (P for "*Pseudo-complement*"),
$D(L)$:= the set of all *complement* elements of L (D for "*Direct summand*").

We introduced and studied in [10] and [9] the conditions (C_1) and (C_{11}) on lattices as the latticial counterparts of the known corresponding conditions on modules. Specifically, for a lattice L one may consider the following conditions.

(C_1) For every $x \in L$ there exists $d \in D(L)$ such that $x \in E(d/0)$.
(C_{11}) For every $x \in L$ there exists a pseudo-complement p of x with $p \in D(L)$.

L is said to be a *CC* or *extending* lattice (respectively, a C_{11} lattice) if it satisfies the condition (C_1) (respectively, (C_{11})).

Next, we recall from [5] the following concept. A mapping $f : L \longrightarrow L'$ between a lattice L with least element 0 and greatest element 1 and a lattice L' with least element

$0'$ and greatest element $1'$ is called a *linear morphism* if there exist $k \in L$, called a *kernel* of f, and $a' \in L'$ such that the following two conditions are satisfied.

- $f(x) = f(x \vee k)$, $\forall\, x \in L$.
- f induces a lattice isomorphism

$$\bar{f} : 1/k \xrightarrow{\sim} a'/0',\ \bar{f}(x) = f(x),\ \forall\, x \in 1/k.$$

If $f : L \longrightarrow L'$ is a linear morphism of lattices, then, by [5, Proposition 1.3] f is an increasing mapping, commutes with arbitrary joins (i.e., $f\big(\bigvee_{i\in I} x_i\big) = \bigvee_{i\in I} f(x_i)$ for any family $(x_i)_{i\in I}$ of elements of L, provided both joins exist), preserves intervals (i.e., for any $u \leqslant v$ in L, one has $f(v/u) = f(v)/f(u)$), and its kernel k is uniquely determined.

As in [7, Examples 0.2 (2)], for any lattice L and any a, b in L such that $a \wedge b = 0$, the mapping

$$q : (a \vee b)/0 \longrightarrow a/0,\ q(x) := (x \vee b) \wedge a,$$

called the *canonical projection* on $a/0$, is a surjective linear morphism with kernel b. Observe that if the lattice L is modular, then

$$q(x) = x,\ \forall\, x \in a/0.$$

Notice that the linear morphism q is the latticial counterpart of the canonical projection $M \oplus M' \longrightarrow M$ for two modules M_R and M'_R.

As in [5], the class $\mathcal{M}$ of all (bounded) modular lattices becomes a category, denoted by $\mathcal{LM}$ ($\mathcal{L}$ for "Linear" and $\mathcal{M}$ for "Modular") if for any $L, L' \in \mathcal{M}$ one takes as morphisms from L to L' all the linear morphisms from L to L'.

The isomorphisms in the category $\mathcal{LM}$ are exactly the isomorphisms in the full category $\mathcal{M}$ of the category $\mathcal{L}$ of all (bounded) lattices. The monomorphisms (respectively, epimorphisms) in the category $\mathcal{LM}$ are exactly the injective (respectively, surjective) linear morphisms. Moreover, the subobjects of $L \in \mathcal{LM}$ can be viewed as the intervals $a/0$ for any $a \in L$. For all these properties of the category $\mathcal{LM}$, see [5, Proposition 2.2].

As in [7], a *lattice preradical* is any functor $r : \mathcal{LM} \longrightarrow \mathcal{LM}$ satisfying the following two conditions.

- For any $L \in \mathcal{LM}$, $r(L)$ is a subobject of L, i.e., an interval $[0, a]$, $a \in L$.
- For any morphism $f : L \longrightarrow L'$ in $\mathcal{LM}$, $r(f) : r(L) \longrightarrow r(L')$ is the restriction of f to $r(L)$ and $r(L')$, i.e., $f(r(L)) \subseteq r(L')$.

In other words, a lattice preradical is nothing else than a subfunctor of the identity functor $1_{\mathcal{LM}}$ of the category $\mathcal{LM}$. More generally, one can define a lattice preradical on any full subcategory $\mathcal{C}$ of $\mathcal{LM}$ as a subfunctor of the identity functor $1_{\mathcal{C}}$ of $\mathcal{C}$.

Throughout this paper R will denote an associative ring with non-zero identity element, and Mod-R the category of all unital right R-modules. The notation M_R will be used to designate a unital right R-module M, and $N \leqslant M$ will mean that N is a submodule of M. The lattice of all submodules of a module M_R will be denoted by $\mathcal{L}(M_R)$.

1. Fully invariant-extending modular lattices

In this section we introduce and investigate the latticial counterpart of the concept of a fully invariant-extending module defined in [12] as follows: a module M is said to be *fully invariant-extending,* shortly *FI-extending,* if every fully invariant submodule of M is essential in a direct summand of M. The concept of a FI-extending module generalizes the concept of an extending (or CS) module by asking that only every fully invariant submodule is essential in a direct summand rather than every submodule.

Throughout this paper, for any $L \in \mathcal{M}$, we denote by $\mathrm{End}(L)$ the monoid of all linear morphisms from L to L.

Definitions 1.1. Let $L \in \mathcal{M}$. An element $a \in L$ is called *fully invariant,* abbreviated *FI,* if $f(a) \leqslant a$ for any $f \in \mathrm{End}(L)$, and the set of all fully invariant elements of L will be denoted by $FI(L)$.

The lattice L is said to be *fully invariant-extending,* shortly *FI-extending,* if for every $x \in FI(L)$ there exists $d \in D(L)$ such that $x \in E(d/0)$. □

Examples 1.2. (1) For any complete modular lattice L, the socle $\mathrm{Soc}\,(L)$, the radical $\mathrm{Rad}\,(L)$, and the Jacobson radical $\mathrm{Jac}\,(L)$ of L are fully invariant elements of L, as these follow from their properties (see, e.g., [6], [8], and/or [7]). Notice that each of Soc, Rad, and Jac is a preradical on the full subcategory $\mathcal{LM}_c$ of $\mathcal{LM}$ consisting of all complete modular lattices.

(2) More generally, for any lattice preradical r on $\mathcal{LM}$, $r(L)$ is a fully invariant element of L for any $L \in \mathcal{M}$, by the definition of a lattice preradical. □

Proposition 1.3. *The following assertions are equivalent for $L \in \mathcal{M}$.*

(1) *L is FI-extending.*
(2) *For every $x \in FI(L)$ there exists a pseudo-complement p of x such that $p \in D(L)$.*

Proof. (1) $\Longrightarrow$ (2) Let $x \in FI(L)$. By definition, there exists $d \in D(L)$ with $x \in E(d/0)$. Let $d' \in L$ be such that $d \vee d' = 1$ and $d \wedge d' = 0$, and let

$$\pi : L = 1/0 = (d \vee d')/0 \longrightarrow d/0,\ \pi(a) := (a \vee d') \wedge d,\ \forall\, a \in L,$$

be the canonical projection on $d/0$. Then, d' is the desired pseudo-complement of x, i.e., d' is maximal in the set of all $c \in L$ with $x \wedge c = 0$. Indeed, $x \wedge d' \leqslant d \wedge d' = 0$, and

if $d' \leqslant b$ with $x \wedge b = 0$, then $x \wedge (d \wedge b) = 0$, so $d \wedge b = 0$ because $x \in E(d/0)$. By modularity, we have

$$b = 1 \wedge b = (d \vee d') \wedge b = (d \wedge b) \vee d' = 0 \vee d' = d',$$

which shows that d' is a pseudo-complement of x, as desired.

(2) $\Longrightarrow$ (1) Let $x \in FI(L)$. By assumption, there exists a pseudo-complement p of x with $p \in D(L)$. Then there exists $p' \in L$ such that $p \vee p' = 1$ and $p \wedge p' = 0$. Let

$$\pi' : L = 1/0 = (p \vee p')/0 \longrightarrow p/0,\ \pi'(a) := (a \vee p') \wedge p,\ \forall\, a \in L,$$

be the canonical projection on $p/0$. Because $x \in FI(L)$, we have $\pi'(x) \leqslant x \wedge p = 0$, so $(x \vee p') \wedge p = 0$. Then $x \leqslant p'$ because, by modularity,

$$x \vee p' = (x \vee p') \wedge 1 = (x \vee p') \wedge (p \vee p') = ((x \vee p') \wedge p) \vee p' = 0 \vee p' = p'.$$

We claim that $x \in E(p'/0)$. In order to prove our claim, observe that, because p is a pseudo-complement of x, we have $p \vee x \in E(L)$ (see, e.g., [4, Lemma 1.2.15]).

Let $b \leqslant p'$ with $b \wedge x = 0$. We have to show that $b = 0$. Now, $(x \vee b) \wedge p \leqslant p' \wedge p = 0$, which implies that $b \wedge (p \vee x) = 0$ by [4, Lemma 1.2.6], and then $b = 0$ because $p \vee x \in E(L)$. This proves our claim. □

Proposition 1.4. *Consider the following assertions for an essentially closed lattice* $L \in \mathcal{M}$.

(1) L *is FI-extending.*
(2) $C(L) \cap FI(L) \subseteq D(L)$.
(3) $C(L) \cap FI(L) = D(L) \cap FI(L)$.

Then (1) $\Longrightarrow$ (2) *and* (2) $\Longleftrightarrow$ (3).

Proof. (1) $\Longrightarrow$ (2) Assume that L is FI-extending. Let $x \in C(L) \cap FI(L)$. Then there exists $d \in D(L)$ such that $x \in E(d/0)$, and hence $x = d \in D(L)$ because L is essentially closed. So, $C(L) \cap FI(L) \subseteq D(L)$.

(2) $\Longleftrightarrow$ (3) This is clear because, by [10, Proposition 1.7 (1)], $D(L) \subseteq C(L)$. □

Proposition 1.5. *Consider the following assertions for a strongly pseudo-complemented lattice* $L \in \mathcal{M}$.

(1) L *is FI-extending.*
(2) $P(L) \cap FI(L) \subseteq D(L)$.
(3) $P(L) \cap FI(L) = D(L) \cap FI(L)$.

Then (1) $\Longrightarrow$ (2) *and* (2) $\Longleftrightarrow$ (3).

Proof. (1) $\Longrightarrow$ (2) By [10, Proposition 1.7 (1)], $P(L) \subseteq C(L)$, so $P(L) \cap FI(L) \subseteq C(L) \cap FI(L)$. Notice that a strongly pseudo-complemented lattice is essentially closed by [4, Theorem 1.2.24]. Hence $P(L) \cap FI(L) \subseteq D(L)$ by Proposition 1.4.

(2) $\Longleftrightarrow$ (3) This is clear, because $D(L) \subseteq P(L)$ by [10, Proposition 1.7 (1)]. □

Remarks 1.6. (1) Observe that if the implication (2) $\Longrightarrow$ (1) in Proposition 1.4 for an essentially closed lattice $L \in \mathcal{M}$ is true, then so is also (2) $\Longrightarrow$ (1) in Proposition 1.5 for a strongly pseudo-complemented lattice $L \in \mathcal{M}$. Indeed, let $x \in FI(L)$ and assume that $P(L) \cap FI(L) \subseteq D(L)$. Then, by [4, Theorem 1.2.24], L is an E-complemented lattice, and hence $P(L) = C(L)$ by [4, Corollary 1.2.17]. By assumption, $C(L) \cap FI(L) \subseteq D(L)$ for an essentially closed lattice L. Thus L is FI-extending by the assumption in Proposition 1.4.

(2) We have no counterexamples for the implications (2) $\Longrightarrow$ (1) in Proposition 1.4 and Proposition 1.5, but we guess that they are *not true* in general.

However, for the lattice $\mathcal{L}(M)$ of all submodules of a right module M, the implication (2) $\Longrightarrow$ (1) in Proposition 1.4 holds when the module M is nonsingular, or more generally, non-M-singular. Indeed, by [13, Lemma 1.4], for such a module M, the essential closure N^c of any FI submodule N of M is again a FI submodule of M. Since N is an essential submodule of the essentially closed submodule N^c of M, and $\mathcal{L}(M)$ is an essentially closed modular lattice, N^c is a direct summand of M in case we assume that then condition (2) holds. It follows that M is a FI-extending module, i.e., the implication (2) $\Longrightarrow$ (1) holds for this lattice $\mathcal{L}(M)$. The same happens for the lattice $\mathcal{L}(R_R)$ of right ideals of a semiprime ring R. Indeed, in view of [11, Lemma 2.2] and [14, Theorem 3.2.37], R is a quasi-Baer ring if and only if every right ideal of R which is essentially closed in R is a direct summand of R, if and only if the module R_R is FI-extending. With similar arguments as in the previous situation, we deduce that the implication (2) $\Longrightarrow$ (1) holds for the lattice $\mathcal{L}(R_R)$. □

The next result shows that the C_{11} lattices lie strictly between the CC lattices and the FI-extending lattices.

Proposition 1.7. *Any C_{11} lattice is FI-extending, but not conversely.*

Proof. Let L be a C_{11} lattice, and let $x \in FI(L)$. Then, by definition, there exists a pseudo-complement p of x with $p \in D(L)$, and so, by Proposition 1.3, it follows that L is FI-extending.

An example of a FI-extending lattice that is not a C_{11} lattice is the lattice of $\mathcal{L}(D_D)$ of all right ideals of a domain D which is not right Ore (see [26, Proposition 4.67]). □

Lemma 1.8. *The following assertions hold for a lattice $L \in \mathcal{M}$.*

(1) *$FI(L)$ is a sublattice of L. Moreover, if the lattice L is also complete, then $FI(L)$ is a complete sublattice of L.*

(2) *If $a \leqslant b$ are elements of L such that $b \in FI(L)$ and $a \in FI(b/0)$, then $a \in FI(L)$.*
(3) *If $a \in FI(L)$ and $d \in D(L)$, then $a \wedge d \in FI(d/0)$.*
(4) *If $a \in FI(L)$ and $(a_j)_{j\in J}$ is a finite independent family of elements of L such that 1 is its direct join $1 = \dot{\bigvee}_{j\in J} a_j$, then*

$$a = \dot{\bigvee}_{j\in J} (a \wedge a_j) \ \text{ and } \ a \wedge a_j = p_j(a), \forall\, j \in J,$$

where, for any $j \in J$, p_j is the canonical projection on $a_j/0$.
(5) *The property in (4) holds for any infinite set J in case L is also upper continuous.*

Proof. (1) Assume that L is a complete lattice, and let $(x_j)_{j\in J}$ be a family of elements in $FI(L)$. Then, for any $f \in \mathrm{End}(L)$ we have $f(x_j) \leqslant x_j$, $\forall\, j \in J$, so

$$f\big(\bigvee_{j\in J} x_j\big) = \bigvee_{j\in J} f(x_j) \leqslant \bigvee_{j\in J} x_j$$

because f commutes with arbitrary joins, and hence $\bigvee_{j\in J} x_j \in FI(L)$. Now, for any $i \in J$, we have

$$f\big(\bigwedge_{j\in J} x_j\big) \leqslant f(x_i) \leqslant x_i$$

because f is an increasing mapping. Then $f\big(\bigwedge_{j\in J} x_j\big) \leqslant \bigwedge_{i\in J} x_i$, which shows that $\bigwedge_{j\in J} x_j \in FI(L)$. Thus, $FI(L)$ is a complete sublattice of L. Similarly, $FI(L)$ is a sublattice of L without any additional condition on L.

(2) Let $f \in \mathrm{End}(L)$. Then $f(b) \leqslant b$, and consider the restriction $f|_{b/0} : b/0 \longrightarrow b/0$ of f to $b/0$, which clearly belongs to $\mathrm{End}(b/0)$. Therefore

$$f|_{b/0}(a) = f(a) \leqslant a,$$

i.e., $a \in FI(L)$, as desired.

(3) There exists $d' \in L$ such that $d \vee d' = 1$ and $d \wedge d' = 0$. Let

$$p : L = 1/0 = (d \vee d')/0 \longrightarrow d/0,\ p(x) := (x \vee d') \wedge d,\ \forall\, x \in L,$$

be the canonical projection on $d/0$. As noticed in Preliminaries, by modularity, we have

$$p(x) = x,\ \forall\, x \in d/0.$$

Let $f \in \mathrm{End}(d/0)$. We have to prove that $f(a\wedge d) \leqslant a\wedge d$. Denote by g the composition of the linear morphisms

$$L \xrightarrow{\ p\ } d/0 \xrightarrow{\ f\ } d/0 \xrightarrow{\ \iota\ } L,$$

where $\iota : d/0 \hookrightarrow L$ is the canonical inclusion mapping, i.e.,

$$g = \iota \circ f \circ p.$$

We have

$$g(a \wedge d) = (\iota \circ f \circ p)(a \wedge d) = f(p(a \wedge d)) = f(a \wedge d) \leqslant f(d) \leqslant d$$

since the linear morphism $f : d/0 \longrightarrow d/0$ is increasing.

Now, $g(a \wedge d) \leqslant g(a) \leqslant a$ because $a \in FI(L)$. We conclude that

$$f(a \wedge d) = g(a \wedge d) \leqslant a \wedge d,$$

as desired.

(4) We proceed by induction on the number n of elements of J. Clearly, we may assume that $J = \{1, \ldots, n\}$. If $n = 1$, we have nothing to prove. For $n = 2$, as in the proof of [7, Proposition 1.3], since $(a \vee a_1) \wedge a_2 \leqslant a_2 \leqslant a \vee a_2$, we deduce by modularity that

$$\begin{aligned}((a \vee a_2) \wedge a_1) \vee ((a \vee a_1) \wedge a_2) = (a \vee a_2) \wedge (a_1 \vee ((a \vee a_1) \wedge a_2)) = \\ = (a \vee a_2) \wedge ((a_1 \vee a_2) \wedge (a \vee a_1)) = (a \vee a_1) \wedge (a \vee a_2),\end{aligned}$$

and so, because $a \in FI(L)$, we deduce that

$$a \leqslant ((a \vee a_2) \wedge a_1) \vee ((a \vee a_1) \wedge a_2) = p_1(a) \vee p_2(a) \leqslant a.$$

It follows that

$$a = p_1(a) \vee p_2(a).$$

Since

$$p_1(a) \leqslant a \wedge a_1 \leqslant (a \vee a_2) \wedge a_1 = p_1(a) \text{ and } p_2(a) \leqslant a \wedge a_2 \leqslant (a \vee a_1) \wedge a_2 = p_2(a),$$

we get

$$p_1(a) = a \wedge a_1 \text{ and } p_2(a) = a \wedge a_2,$$

and consequently

$$a = (a \wedge a_1) \vee (a \wedge a_2),$$

as desired.

If $n \geqslant 2$, assume that the result is true for n. We will prove it for $n+1$. Set $b_n := \bigvee_{1 \leqslant j \leqslant n} a_j$. We have $1 = b_n \dot{\vee} a_{n+1}$, so

$$a = (a \wedge b_n) \vee (a \wedge a_{n+1})$$

by the case $n = 2$ we have just proved above.

Since $b_n \in D(L)$, $a \wedge b_n \in FI(b_n/0)$ by (3). By the inductive hypothesis for the lattice $b_n/0$, we have

$$a \wedge b_n = \bigvee_{1 \leqslant j \leqslant n} ((a \wedge b_n) \wedge a_j) = \bigvee_{1 \leqslant j \leqslant n} (a \wedge a_j),$$

and so

$$a = \bigvee_{1 \leqslant j \leqslant n+1} (a \wedge a_j),$$

as desired.

As in the case $n = 2$, for any $j \in J = \{1, \ldots, n\}$, we have

$$p_j(a) \leqslant a \wedge a_j \leqslant (a \vee b_j) \wedge a_j = p_j(a),$$

where $b_k := \bigvee_{j \in J \setminus \{k\}} a_j$, $\forall\, k \in J$, so

$$p_j(a) = a \wedge a_j, \forall\, j \in J.$$

(5) Denote by $\mathcal{P}_f(J)$ the upward directed set of all finite subsets of J. For any $F \in \mathcal{P}_f(J)$ set $a_F := \bigvee_{j \in F} a_j$. Because $a_F \in D(L)$, $a \wedge a_F \in FI(a_F/0)$ by (3), so using (4) and the upper continuity of L, we obtain

$$\bigvee_{j \in J} (a \wedge a_j) = \bigvee_{F \in \mathcal{P}_f(J)} \big(\bigvee_{j \in F} (a \wedge a_j) \big) = \bigvee_{F \in \mathcal{P}_f(J)} \big(\bigvee_{j \in F} ((a \wedge a_F) \wedge a_j) \big) =$$

$$= \bigvee_{F \in \mathcal{P}_f(J)} (a \wedge a_F) = a \wedge \big(\bigvee_{F \in \mathcal{P}_f(J)} a_F \big) = a \wedge \big(\bigvee_{j \in J} a_j \big) = a \wedge 1 = a,$$

and we are done.

Now the equality

$$p_j(a) = a \wedge a_j, \forall\, j \in J,$$

follows as in (4). □

Proposition 1.9. *Let $L \in \mathcal{M}$ and $a \in FI(L)$. If L is FI-extending, then so is also $a/0$.*

Proof. Let $b \in FI(a/0)$. Then $b \in FI(L)$ by Lemma 1.8 (2), so there exists $d \in D(L)$ such that $b \in E(d/0)$. Let $d' \in L$ with $d \vee d' = 1$ and $d \wedge d' = 0$. Let

$$p : L \longrightarrow d/0,\ p(x) := (x \vee d') \wedge d,\ \forall\, x \in L,$$

be the canonical projection on $d/0$.

Then

$$b = p(b) \leqslant p(a) \wedge d = p(a) \leqslant d,$$

and so $b \in E(p(a)/0)$. By the proof of Lemma 1.8 (4) for $n = 2$,

$$a = p(a) \dot{\vee} p'(a),$$

where p' is the canonical projection on $d'/0$, so $p(a) \in D(a/0)$ and $b \in E(p(a)/0)$, which completes the proof. □

2. Inheritance of the FI-extending condition under direct joins

The condition (C_1) is, in general, not inherited by direct joins, as is well-known for modules (see, e.g., [17]). This contrasts with the condition (C_{11}) by [10, Theorem 2.6]. In this section we prove that the weaker FI-extending condition for modular lattices is inherited by direct joins and also discuss whether this condition is inherited under complement intervals.

Theorem 2.1. *Let $L \in \mathcal{M}$, and let $(a_j)_{1\leqslant j\leqslant n}$ be a finite independent family of elements of L such that $1 = \dot{\bigvee}_{1\leqslant j\leqslant n} a_j$ and $a_i/0$ is FI-extending for all $1 \leqslant i \leqslant n$. Then L is a FI-extending lattice.*

Proof. Denote by p_j the canonical projection on $a_j/0$, $1 \leqslant j \leqslant n$, and let $a \in FI(L)$. Since $p_j(a) = a \wedge a_j \in FI(a_j/0)$ for every j, $1 \leqslant j \leqslant n$, with $p_j(a) \neq 0$, there exists $d_j \in D(a_j/0)$ such that $p_j(a) \in E(d_j/0)$. By Lemma 1.8 (4), $a = \dot{\bigvee}_{1\leqslant j\leqslant n} p_j(a)$, so $a \in E\big((\dot{\bigvee}_{1\leqslant j\leqslant n} d_j)/0\big)$. But $\dot{\bigvee}_{1\leqslant j\leqslant n} d_j \in D(L)$, and we are done. □

Corollary 2.2. *Let $L \in \mathcal{M}$, and let $(a_i)_{1\leqslant i\leqslant n}$ be a finite independent family of elements of L such that $1 = \dot{\bigvee}_{1\leqslant i\leqslant n} a_i$. If $a_i/0$ is an extending (in particular, a uniform) lattice for all i, $1 \leqslant i \leqslant n$, then L is FI-extending.*

Proof. The result follows at once from Theorem 2.1 because any extending lattice is clearly FI-extending. □

Theorem 2.3. *Let L be an upper continuous modular lattice, and let $(a_i)_{i\in I}$ be an independent family of elements of L such that $1 = \dot{\bigvee}_{j\in J}\ a_j$ and $a_i/0$ is FI-extending for all $i \in I$. Then L is a FI-extending lattice.*

Proof. Proceed as in Theorem 2.1 by using Lemma 1.8 (5) instead of Lemma 1.8 (4). □

Corollary 2.4. *Let L be an upper continuous modular lattice, and let $(a_i)_{i\in I}$ be an independent family of elements of L such that $1 = \dot{\bigvee}_{j\in J}\ a_j$ and $a_i/0$ is an extending (in particular, a uniform) lattice for all $i \in I$. Then L is a FI-extending lattice.*

Proof. The result follows at once from Theorem 2.3 because any extending lattice is clearly FI-extending. □

The next result is an extension of [12, Corollary 1.5] from Abelian groups to modules over a Dedekind domain.

Corollary 2.5. *Let M be a module over a Dedekind domain D satisfying one of the following three conditions.*

(i) *M is finitely generated.*
(ii) *M is a torsion module of bounded order (i.e., $IM = 0$ for some non-zero ideal I of D).*
(iii) *M is divisible (i.e., $M = rM$ for any $r \in D \setminus \{0\}$).*

Then M is FI-extending.

Proof. Of course we may assume that D is not a field. Any cyclic D-module is either isomorphic to D or to D/I for some nonzero ideal I of D.

In the first case, D is a uniform D-module because $J \cap K \neq 0$ for any two nonzero ideals J and K of D, and any nonzero ideal of D is a uniform D-module too.

In the second case, write I as a finite product of powers of maximal ideals of D, and then apply the Chinese Remainder Lemma to express D/I as a module isomorphic to a finite direct sum of cyclic modules of type $D/\mathfrak{m}^n$ for $\mathfrak{m}$ a maximal ideal of D and n a positive integer. Now, observe that each $D/\mathfrak{m}^n$ is a uniform module because $\mathfrak{m}^n$ is an irreducible ideal of D.

(i) Since D is a Noetherian ring, any finitely generated D-module is Noetherian. By [20, Theorem 1], the torsion submodule T of M is a direct summand of M, and M/T is isomorphic to a (finite) direct sum of nonzero ideals of D; moreover the torsion module T is isomorphic to a (finite) direct sum of cyclic modules, and as noticed above, each of these cyclic modules is a uniform D-module. Apply now Corollary 2.2.

(ii) By [18, Theorem 1], every torsion D-module of bounded order is isomorphic to a direct sum of cyclic modules. Hence M is again a direct sum of uniform D-modules, and so, a FI-extending module.

(iii) By [20, Theorem 6], M is a divisible D-module if and only if M is an injective D-module, so M is clearly an extending D-module. □

Example 2.6. (Example 1.6 [12]) Let

$$M := \prod_{p>0 \text{ prime}} \mathbb{Z}/p\mathbb{Z}.$$

The torsion subgroup $t(M)$ of M is fully invariant and essentially closed, but $t(M)$ is not a direct summand of M, and therefore M is not FI-extending. This example also shows that a direct product of FI-extending modules need not be a FI-extending module. We do not know whether a direct product of a family of FI-extending lattices is also FI-extending.

Remark 2.7. It is an open problem if the concept of a FI-extending lattice is closed under complement intervals, i.e., if L is a FI-extending lattice then does it follow that so is $d/0$ for any $d \in D(L)$? By [14, p. 45] and [23] the answer is not known even for modules over arbitrary rings. □

The next result provides a case when the inheritance of the FI-extending condition under complement intervals does hold.

Proposition 2.8. *Let L be a modular FI-extending lattice such that $1 = a \dot{\vee} b$ and $a \in FI(L)$. Then $a/0$ and $b/0$ are both FI-extending.*

Proof. By Proposition 1.9, $a/0$ is FI-extending. In order to prove that $b/0$ is FI-extending, let $c \in FI(b/0)$. We claim that $a \dot{\vee} c \in FI(L)$, i.e., $g(a \vee c) \leqslant a \vee c$, $\forall\, g \in \mathrm{End}(L)$.

For a fixed $g \in \mathrm{End}(L)$, denote by h the composition

$$b/0 \xrightarrow{\iota} L \xrightarrow{g} L \xrightarrow{p} b/0,$$

where $\iota : b/0 \hookrightarrow L$ is the canonical inclusion mapping and $p : L \longrightarrow b/0$ is the canonical projection on $b/0$. Then $h \in \mathrm{End}(b/0)$ as a composition of linear morphisms of lattices.

We have

$$h(x) = (p \circ g \circ \iota)(x) = p(g(x)) = (g(x) \vee a) \wedge b,\ \forall\, x \in b/0.$$

Now observe that $h(c) \leqslant c$, because $c \in FI(b/0)$ and $h \in \mathrm{End}(b/0)$. It follows that

$$h(c) = (g(c) \vee a) \wedge b \leqslant c.$$

We will prove now our claim. Since $1 = a \dot{\vee} b$, we have

$$g(c) \vee a = (g(c) \vee a) \wedge (a \vee b) = a \vee ((g(c) \vee a) \wedge b) = a \vee h(c) \leqslant a \vee c$$

by modularity. Using the properties presented in Section 0 of linear morphisms of lattices, we have

$$g(a \vee c) = g(a) \vee g(c) \leqslant a \vee g(c) \leqslant a \vee c,$$

and we are done.

Thus $a \dot{\vee} c \in FI(L)$, so there exists $d \in D(L)$ with $a \dot{\vee} c \in E(d/0)$. Let $d' \in L$ be with $1 = d \dot{\vee} d'$. From $1 = a \dot{\vee} b$, we deduce $d = a \dot{\vee} (d \wedge b)$ by modularity. Since $a \dot{\vee} c \leqslant d$ and $c \leqslant b$, then $c \leqslant d \wedge b$. Now $a \dot{\vee} c \in E(d/0)$ entails $(a \dot{\vee} c) \wedge b \in E((d \wedge b)/0)$. Observe that, by modularity

$$(a \dot{\vee} c) \wedge b = (a \wedge b) \vee c = 0 \vee c = c,$$

so $c \in E((d \wedge b)/0)$.

Clearly $1 = d \dot{\vee} d' = (d \wedge b) \vee a \vee d'$, and then, again by modularity, we have

$$b = (d \wedge b) \dot{\vee} ((a \dot{\vee} d') \wedge b).$$

This shows that $c \in E((d \wedge b)/0)$ and $(d \wedge b) \in D(b/0)$, i.e., $b/0$ is FI-extending, as desired. □

Corollary 2.9. *Let L be an upper continuous modular lattice, and let $(a_j)_{j \in J}$ be an independent family of elements of L such that $1 = \dot{\bigvee}_{j \in J} a_j$. If $a_j/0$ is an extending (in particular, a uniform) lattice for all $j \in J$, then L is FI-extending.*

Proof. As in Corollary 2.2. □

As in [1], a lattice L is said to be RU (for Rich in Uniforms) if $L \neq \{0\}$ and, for any $0 \neq x \in L$, there exists a uniform element $u \in L$ with $u \leqslant x$. Clearly, any modular lattice with finite Goldie dimension is RU, and, by [1, Examples 1.3], any lattice having Gabriel dimension, in particular, any semi-Artinian lattice or lattice having (dual) Krull dimension, is RU.

Corollary 2.10. *Any modular RU lattice contains an essential FI-extending sublattice.*

Proof. Let L be a modular RU lattice. By [1, Theorem 1.16], L has an independent set U of uniform elements that is essential in L, so $\bigvee U \in E(L)$. By Corollary 2.9, $\left(\bigvee U\right)/0$ is a FI-extending sublattice of L. □

Remark 2.11. In the case of modules, any module M is can be embedded into a FI-extending module, namely the injective hull $E(M)$ of M. It is not clear whether a modular lattice can be embedded into a FI-extending lattice. Notice that, by [5, 4.7], there is no non-trivial linear injective lattice.

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References

[1] T. Albu, Completely irreducible meet decompositions in lattices, with applications to Grothendieck categories and torsion theories (I), Bull. Math. Soc. Sci. Math. Roumanie 52 (100) (2009) 393–419.
[2] T. Albu, The Osofsky–Smith theorem for modular lattices, and applications (I), Comm. Algebra 39 (2011) 4488–4506.
[3] T. Albu, The Osofsky–Smith theorem for modular lattices, and applications (II), Comm. Algebra 42 (2014) 2663–2683.
[4] T. Albu, Chain Conditions in Modular Lattices with Applications to Grothendieck Categories and Torsion Theories, Monograph Series of Parana's Mathematical Society No. 1, Sociedade Paranaense de Matemática, Maringá, Paraná, Brasil, 2015, 134 pp.
[5] T. Albu, M. Iosif, The category of linear modular lattices, Bull. Math. Soc. Sci. Math. Roumanie 56 (104) (2013) 33–46.
[6] T. Albu, M. Iosif, On socle and radical of modular lattices, Ann. Univ. Buchar. Math. Ser. 5 (LXIII) (2014) 187–194.
[7] T. Albu, M. Iosif, Lattice preradicals with applications to Grothendieck categories and torsion theories, J. Algebra 444 (2015) 339–366.
[8] T. Albu, M. Iosif, Lattice preradicals versus module preradicals, Ann. Univ. Buchar. Math. Ser. 6 (LXIV) (2015) 19–34.
[9] T. Albu, M. Iosif, New results on C_{11} and C_{12} lattices, with applications to Grothendieck categories and torsion theories, Front. Math. China 11 (2016) 815–828.
[10] T. Albu, M. Iosif, A. Tercan, The conditions (C_i) in modular lattices, and applications, J. Algebra Appl. 15 (2016) 1650001 [19 pages].
[11] G.F. Birkenmeier, A generalization of FPF rings, Comm. Algebra 17 (1989) 855–884.
[12] G.F. Birkenmeier, B.J. Müller, S.T. Rizvi, Modules in which every fully invariant submodule is essential in a direct summand, Comm. Algebra 30 (2002) 1395–1415.
[13] G.F. Birkenmeier, J.K. Park, S.T. Rizvi, Modules with fully invariant submodules essential in fully invariant summands, Comm. Algebra 30 (2002) 1833–1852.
[14] G.F. Birkenmeier, J.K. Park, S.T. Rizvi, Extensions of Rings and Modules, Birkhäuser/Springer, New York, 2013.
[15] G. Birkhoff, Lattice Theory, 3rd ed., American Mathematical Society, Providence, RI, 1967.
[16] P. Crawley, R.P. Dilworth, Algebraic Theory of Lattices, Prentice-Hall, Englewood Cliffs, New Jersey, 1973.
[17] N.V. Dung, D.V. Huynh, P.F. Smith, R. Wisbauer, Extending Modules, Longman Scientific & Technical, Harlow, 1994.
[18] C. Faith, On Köthe rings, Math. Ann. 164 (1966) 207–212.

[19] G. Grätzer, General Lattice Theory, second edition, Birkhäuser Verlag, Basel Boston Berlin, 2003.
[20] I. Kaplansky, Modules over Dedekind rings and valuation rings, Trans. Amer. Math. Soc. 72 (1952) 327–340.
[21] S.H. Mohamed, B.J. Müller, Continuous and Discrete Modules, Cambridge University Press, Cambridge, 1990.
[22] B. Osofsky, P.F. Smith, Cyclic modules whose quotients have all complement submodules direct summands, J. Algebra 139 (1991) 342–354.
[23] S.T. Rizvi, private communication to T. Albu, March 2017.
[24] P.F. Smith, A. Tercan, Generalizations of CS-modules, Comm. Algebra 21 (1993) 1809–1847.
[25] B. Stenström, Rings of Quotients, Springer-Verlag, Berlin Heidelberg New York, 1975.
[26] A. Tercan, C.C. Yücel, Module Theory, Extending Modules and Generalizations, Birkhäuser, 2016.