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On the spectral radius of bipartite graphs which are nearly complete

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Abstract

For $p,q,r,s,t \in \mathbb{Z}^+$ with $rt \le p$ and $st \le q$, let $G = G(p,q;r,s;t)$ be the bipartite graph with partite sets $U = \{u_1,\dots,u_p\}$ and $V = \{v_1,\dots,v_q\}$ such that any two edges u_i and v_j are not adjacent if and only if there exists a positive integer k with $1 \le k \le t$ such that $(k-1)r+1 \le i \le kr$ and $(k-1)s+1 \le j \le ks$. Under these circumstances, Chen *et al.* (Linear Algebra Appl. 432:606-614, 2010) presented the following conjecture:

Assume that $p \le q$, $k < p$, $|U| = p$, $|V| = q$ and $|E(G)| = pq - k$. Then whether it is true that

$$\lambda_1(G) \le \lambda_1(G(p,q;k,1;1)) = \sqrt{\frac{pq-k+\sqrt{p^2q^2-6pqk+4pk+4qk^2-3k^2}}{2}}.$$

In this paper, we prove this conjecture for the range $\min_{v_h \in V}\{\deg v_h\} \le \lfloor \frac{p-1}{2} \rfloor$.

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1 Introduction

Let G be a (simple) graph with the vertex and edge sets given by $V(G) = \{v_1,v_2,\dots,v_n\}$ and $E(G) = \{v_iv_j \mid v_i \text{ and } v_j \text{ are adjacent}\}$, respectively. The *adjacency matrix* of G on n vertices is an $n \times n$ matrix $A(G)$ whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1; & \text{if } v_iv_j \in E(G), \\ 0; & \text{otherwise.} \end{cases}$$

Since $A(G)$ is symmetric, all the eigenvalues of $A(G)$ are real. In fact, the eigenvalues of $A(G)$ are called *eigenvalues of the graph* G. We can list the eigenvalues of the graph G in a non-increasing order as follows:

$$\lambda_1(G) \ge \lambda_2(G) \ge \cdots \ge \lambda_{n-1}(G) \ge \lambda_n(G).$$

The largest eigenvalue $\lambda_1(G)$ is often called the *spectral radius* of G.

Throughout this paper, we will consider only finite, simple, undirected, bipartite graphs. So, let us suppose that $G = (U \cup V, E)$ is such a bipartite graph, where $U = \{u_1,u_2,\dots,u_p\}$, $V = \{v_1,v_2,\dots,v_q\}$ are two sets of vertices and E is the set of edges defined as a subset of

$U \times V$. As a usual notation, the *degrees* of vertices $u_i \in U$ and $v_j \in V$ will be denoted by $\deg u_i$ and $\deg v_j$, respectively. For the integers $p, q, r, s, t \in \mathbb{Z}^+$ satisfying $rt \le p$ and $st \le q$, let us denote the bipartite graph G by $G(p, q; r, s; t)$ with the above partite sets U and V such that $u_i \in U$ and $v_j \in V$ are *not adjacent* if and only if there exists a $k \in \mathbb{Z}^+$ with $1 \le k \le t$ such that $(k-1)r + 1 \le i \le kr$ and $(k-1)s + 1 \le j \le ks$.

In the literature, upper bounds for the spectral radius in terms of various parameters for unweighted and weighted graphs have been widely investigated [1–10]. As a special case, in [3], Chen *et al.* studied the spectral radius of bipartite graphs which are close to a complete bipartite graph. For partite sets U and V having $|U| = p$, $|V| = q$ and $p \le q$, in the same reference, the authors also gave an affirmative answer to the conjecture [11, Conjecture 1.2] by taking $|E(G)| = pq - 2$ into account of a bipartite graph. Furthermore, refining the same conjecture for the number of edges is at least $pq - p + 1$, there still exists the following conjecture.

Conjecture 1 [3] *For positive integers p, q and k satisfying $p \le q$ and $k < p$, let G be a bipartite graph with partite sets U and V having $|U| = p$ and $|V| = q$, and $|E(G)| = pq - k$. Then*

$$\lambda(G) \le \lambda\bigl(G(p, q; k, 1; 1)\bigr) = \sqrt{\frac{pq - k + \sqrt{p^2q^2 - 6pqk + 4pk + 4qk^2 - 3k^2}}{2}}.$$

We note that similar conjectures in this topic have been resolved by the first author in the papers [12–16]. In here, as the main goal, we present the proof of Conjecture 1 for the range $\min_{v_h \in V}\{\deg v_h\} \le \lfloor \frac{p-1}{2} \rfloor$.

2 Main result

The following lemma will be needed for the proof of our main result.

Lemma 1 [3] *Let λ_1 be the spectral radius of the bipartite graph $G(p, q; k, 1; 1)$. Then*

$$\lambda_1 = \sqrt{\frac{pq - k + \sqrt{p^2q^2 - 6pqk + 4pk + 4qk^2 - 3k^2}}{2}}.$$

We now present an upper bound on the spectral radius of the bipartite graph G.

Theorem 1 *For positive integers p, q and k satisfying $p \le q$ and $k < p$, let G be a bipartite graph with partite sets U and V having $|U| = p$ and $|V| = q$, and $|E(G)| = pq - k$. If $\min_{v_h \in V}\{\deg v_h\} \le \lfloor \frac{p-1}{2} \rfloor$, then*

$$\lambda_1(G) \le \sqrt{\frac{pq - k + \sqrt{p^2q^2 - 6pqk + 4pk + 4qk^2 - 3k^2}}{2}} \tag{1}$$

with equality if and only if $G \cong G(p, q; k, 1; 1)$.

Proof Let $\mathbf{Z} = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)^T$ be an eigenvector of $A(G)$ corresponding to an eigenvalue $\lambda_1(G)$. For the sets U and V, let $x_i = \max_{1 \le h \le p} x_h$ and $y_j = \max_{1 \le h \le q} y_h$, respectively. Also, let us suppose that v_1 is the vertex having minimum degree in V. Then we have

$$\left\lfloor \frac{p-1}{2} \right\rfloor \geq \min_{v_h \in V}\{\deg v_h\} = \deg v_1 = d_1 \quad \text{(say)}.$$

Now,

$$A(G)\mathbf{Z} = \lambda_1(G)\mathbf{Z}. \tag{2}$$

Considering (2), we get

$$\lambda_1(G)x_i \leq (q-1)y_j + y_1 \quad \text{for } u_i \in U \tag{3}$$

and

$$\lambda_1(G)y_1 \leq d_1 x_i \quad \text{for } v_1 \in V. \tag{4}$$

However, from (3) and (4), we clearly obtain

$$\lambda_1^2(G)y_1 \leq d_1\big[(q-1)y_j + y_1\big],$$

which can be written shortly as

$$\big(\lambda_1^2(G) - d_1\big)y_1 \leq (q-1)d_1 y_j. \tag{5}$$

Since v_1 is the vertex with the minimum degree d_1 in V and the total number of edges in bipartite graph G is $pq - k$, we have

$$\sum_{h=1}^{p} \lambda_1(G)x_h \leq (pq - k - d_1)y_j + d_1 y_1. \tag{6}$$

For $v_j \in V$, from (2) we get

$$\lambda_1(G)y_j = \sum_{u_h : u_h v_j \in E} x_h.$$

In other words, by (6),

$$\lambda_1^2(G)y_j = \sum_{u_h : u_h v_j \in E} \lambda_1(G)x_h \leq \sum_{h=1}^{p} \lambda_1(G)x_h \leq (pq - k - d_1)y_j + d_1 y_1,$$

that is,

$$\big(\lambda_1^2(G) - pq + k + d_1\big)y_j \leq d_1 y_1. \tag{7}$$

From (5) and (7), we get

$$\lambda_1^4(G) - (pq - k)\lambda_1^2(G) + d_1(pq - k - qd_1) \leq 0,$$

that is,

$$\lambda_1(G) \leq \sqrt{\frac{pq - k + \sqrt{p^2q^2 - 2pqk + k^2 - 4pqd_1 + 4kd_1 + 4qd_1^2}}{2}}. \tag{8}$$

Let us consider a function

$$f(x) = 4qx^2 + 4kx - 4pqx, \quad \text{where } x \leq \left\lfloor \frac{p-1}{2} \right\rfloor.$$

Then

$$f'(x) = -4q\left(p - \frac{k}{q} - 2x\right) < 0, \quad \text{as } x \leq \left\lfloor \frac{p-1}{2} \right\rfloor \text{ and } k < p \leq q.$$

Thus $f(x)$ is a decreasing function on $1 \leq x \leq \lfloor \frac{p-1}{2} \rfloor$. Since $p - k \leq d_1 \leq \lfloor \frac{p-1}{2} \rfloor$, from (8), we get the required result (1).

Suppose now that equality holds in (1). Then all inequalities in the above argument must become equalities. Thus we have $d_1 = p - k$. From the equality in (3), we get

$$y_h = y_j, \quad h = 2, 3, \ldots, q \quad \text{and}$$
$$u_i v_h \in E, \quad h = 1, 2, \ldots, q.$$

From the equality in (4), we get

$$x_h = x_i, \quad h = p - d_1 + 1, p - d_1 + 2, \ldots, p \quad \text{and}$$
$$u_h v_1 \in E, \quad h = p - d_1 + 1, p - d_1 + 2, \ldots, p.$$

From the equality in (7), we get

$$y_h = y_j, \quad h = 2, 3, \ldots, q \quad \text{and}$$
$$u_h v_j \in E, \quad h = 1, 2, \ldots, p, j = 2, 3, \ldots, q.$$

Hence we conclude that $G \cong G(p, q; k, 1; 1)$.

Conversely, by Lemma 1, one can easily see that the equality holds in (1) for the graph $G(p, q; k, 1; 1)$. □

Remark 1 In Theorem 1, we proved Conjecture 1 for the range $\min_{v_h \in V}\{\deg v_h\} \leq \lfloor \frac{p-1}{2} \rfloor$. However, this conjecture is still open for the range $\lfloor \frac{p-1}{2} \rfloor < \min_{v_h \in V}\{\deg v_h\} < p$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. Moreover, all authors read and approved the final manuscript.

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References
1. Berman, A, Zhang, XD: On the spectral radius of graphs with cut vertices. J. Comb. Theory, Ser. B **83**, 233-240 (2001)
2. Brualdi, RA, Hoffman, AJ: On the spectral radius of a (0, 1) matrix. Linear Algebra Appl. **65**, 133-146 (1985)
3. Chen, YF, Fu, HL, Kim, IJ, Stehr, E, Watts, B: On the largest eigenvalues of bipartite graphs which are nearly complete. Linear Algebra Appl. **432**, 606-614 (2010)
4. Cvetković, D, Doob, M, Sachs, H: Spectra of Graphs. Academic Press, New York (1980)
5. Cvetković, D, Rowlinson, P: The largest eigenvalue of a graph: a survey. Linear Multilinear Algebra **28**, 3-33 (1990)
6. Das, KC, Kumar, P: Bounds on the greatest eigenvalue of graphs. Indian J. Pure Appl. Math. **34**(6), 917-925 (2003)
7. Das, KC, Kumar, P: Some new bounds on the spectral radius of graphs. Discrete Math. **281**, 149-161 (2004)
8. Das, KC, Bapat, RB: A sharp upper bound on the spectral radius of weighted graphs. Discrete Math. **308**, 3180-3186 (2008)
9. Hong, Y: Bounds of eigenvalues of graphs. Discrete Math. **123**, 65-74 (1993)
10. Stanley, RP: A bound on the spectral radius of graphs with e edges. Linear Algebra Appl. **67**, 267-269 (1987)
11. Bhattacharya, A, Friedland, S, Peled, UN: On the first eigenvalue of bipartite graphs. Electron. J. Comb. **15**, #R144 (2008)
12. Das, KC: On conjectures involving second largest signless Laplacian eigenvalue of graphs. Linear Algebra Appl. **432**, 3018-3029 (2010)
13. Das, KC: Conjectures on index and algebraic connectivity of graphs. Linear Algebra Appl. **433**, 1666-1673 (2010)
14. Das, KC: Proofs of conjecture involving the second largest signless Laplacian eigenvalue and the index of graphs. Linear Algebra Appl. **435**, 2420-2424 (2011)
15. Das, KC: Proof of conjectures involving the largest and the smallest signless Laplacian eigenvalues of graphs. Discrete Math. **312**, 992-998 (2012)
16. Das, KC: Proof of conjectures on adjacency eigenvalues of graphs. Discrete Math. **313**(1), 19-25 (2013)

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