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ON A SUBCLASS OF CERTAIN CONVEX HARMONIC FUNCTIONS

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Abstract. We define and investigate a subclass of complex valued harmonic convex functions that are univalent and sense preserving in the open unit disk. We obtain coefficient conditions, extreme points, distortion bounds, convolution conditions for the above family of harmonic functions.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a domain $\mathfrak{D} \subset \mathbb{C}$ if both u and v are real harmonic in $\mathfrak{D}$. In any simply connected domain we can write $f = h+\bar{g}$, where h and g are analytic in $\mathfrak{D}$. A necessary and sufficient condition for f to be locally univalent and sense preserving in $\mathfrak{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathfrak{D}$.

Denote by S_H the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1}$$

In 1984 Clunie and Sheil-Small [3] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses.

We let $SC_H(\lambda, \alpha)$ denote the subclass of S_H consisting of $f = h + \bar{g}$ of the form (1) that satisfy the condition

$$\operatorname{Re}\left\{ \frac{\lambda(z^3h'''(z)-\overline{z^3g'''(z)})+(2\lambda+1)z^2h''(z)+(1-4\lambda)\overline{z^2g''(z)}+zh'(z)+(1-2\lambda)\overline{zg'(z)}}{\lambda(z^2h''(z)+\overline{z^2g''(z)})+zh'(z)+(2\lambda-1)\overline{zg'(z)}} \right\} > \alpha, \tag{2}$$

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where $0 \leq \lambda \leq \alpha/(1+\alpha)$ or $\lambda \geq 1/(1+\alpha)$ and $0 \leq \alpha < 1$.

We further let $SC_{\overline{H}}(\lambda, \alpha)$ denote the subclass of $SC_H(\lambda, \alpha)$ consisting of functions $f = h + \bar{g}$ such that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad g(z) = \sum_{n=1}^{\infty} |b_n| z^n. \tag{3}$$

Recently, Avcı and Zlotkiewicz [2], Jahangiri [4, 5] and Silverman [7], studied the harmonic convex functions. Jahangiri [4] proved that if $f = h + \bar{g}$ is given by (1) and if

$$\sum_{n=1}^{\infty} \left(\frac{n(n-\alpha)}{1-\alpha}|a_n| + \frac{n(n+\alpha)}{1-\alpha}|b_n| \right) \leq 2, \quad 0 \leq \alpha < 1,\ a_1 = 1,$$

then f is harmonic, univalent, and convex of order α in U. This condition is proved to be also necessary if h and g are of the form (3). Avcı and Zlotkiewicz [2] proved that the coefficient condition

$$\sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \leq 1,\ b_1 = 0$$

is sufficient for functions $f = h + \bar{g} \in S_H$ to be harmonic convex. Silverman [7] proved that this coefficient condition is also necessary if a_n and b_n, $(n = 2, 3, \ldots)$ in (1) are negative.

We note that $SC_{\overline{H}}(\lambda, \alpha)$ is the generalization of the family of harmonic convex functions of order α by Jahangiri [4, 5].

In this note, we give sufficient coefficient conditions for normalized harmonic functions in $SC_H(\lambda, \alpha)$. These conditions are also shown to be necessary when h has negative and g has positive coefficients. We also obtain extreme points, distortion bounds, some preliminary results concerning neighborhoods, convolutions and convex combinations for $SC_{\overline{H}}(\lambda, \alpha)$.

2. Main results

We begin with a sufficient coefficient condition for functions in SC_H (λ, α).

THEOREM 2.1. *Let $f = h + \bar{g}$ be so that h and g are given by (1). Furthermore, let*

$$\sum_{n=1}^{\infty} \left(\frac{n(n-\alpha)(\lambda n - \lambda + 1)}{1-\alpha}|a_n| + \frac{n(n+\alpha)|\lambda n + \lambda - 1|}{1-\alpha}|b_n| \right) \leq 2, \tag{4}$$

where $a_1 = 1$, $0 \le \alpha < 1$, $0 \le \lambda \le \alpha/(1+\alpha)$ *or* $\lambda \ge 1/(1+\alpha)$. *Then* $f \in SC_H(\lambda, \alpha)$ *and* f *is sense-preserving, univalent harmonic in* U .

Proof. We show that $f \in SC_H(\lambda, \alpha)$. We only need to show that if (4) holds then the condition (2) is satisfied. In view of (1) the condition (2) takes the form

$$\begin{aligned} &\operatorname{Re}\left\{\frac{1-\alpha+\sum_{n=2}^{\infty} n(n-\alpha)(\lambda n-\lambda+1)a_n z^{n-1} - \sum_{n=1}^{\infty} n(n+\alpha)(\lambda n+\lambda-1)b_n \bar{z}^n/z}{1+\sum_{n=2}^{\infty} n(\lambda n-\lambda+1)a_n z^{n-1} + \sum_{n=1}^{\infty} n(\lambda n+\lambda-1)b_n \bar{z}^n/z}\right\} \\ &= \operatorname{Re}\frac{1+A(z)}{1+B(z)}. \end{aligned}$$

Setting

$$\frac{1+A(z)}{1+B(z)} = \frac{1+w(z)}{1-w(z)},$$

we will have $Re\frac{1+A(z)}{1+B(z)} > 0$ if $|w(z)| < 1$,

$$\begin{aligned} w(z) &= \frac{A(z)-B(z)}{2+A(z)+B(z)} \\ &= \frac{-\alpha+\sum_{n=2}^{\infty} n(n-\alpha-1)(\lambda n-\lambda+1)a_n z^{n-1} - \sum_{n=1}^{\infty} n(n+\alpha+1)(\lambda n+\lambda-1)b_n \bar{z}^n/z}{2-\alpha+\sum_{n=2}^{\infty} n(n-\alpha+1)(\lambda n-\lambda+1)a_n z^{n-1} - \sum_{n=1}^{\infty} n(n+\alpha-1)(\lambda n+\lambda-1)b_n \bar{z}^n/z}, \end{aligned}$$

so that

$$|w(z)| < \frac{\alpha+\sum_{n=2}^{\infty} n(n-\alpha-1)(\lambda n-\lambda+1)|a_n| + \sum_{n=1}^{\infty} n(n+\alpha+1)|\lambda n+\lambda-1||b_n|}{2-\alpha-\left(\sum_{n=2}^{\infty} n(n-\alpha+1)(\lambda n-\lambda+1)|a_n| + \sum_{n=1}^{\infty} n(n+\alpha-1)|\lambda n+\lambda-1||b_n|\right)}.$$

This last expression is bounded above by 1 if and only if

$$\sum_{n=2}^{\infty} n(n-\alpha)(\lambda n-\lambda+1)|a_n| + \sum_{n=1}^{\infty} n(n+\alpha)|\lambda n+\lambda-1||b_n| \le 1-\alpha.$$

If $z_1 \ne z_2$, then for $\lambda \ge 1/(1+\alpha)$ or $0 \le \lambda \le \alpha/(1+\alpha)$

$$\begin{aligned} \left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\ge 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\ &= 1 - \left|\frac{\sum_{n=1}^{\infty} b_n(z_1^n - z_2^n)}{(z_1-z_2)+\sum_{n=2}^{\infty} a_n(z_1^n - z_2^n)}\right| \\ &> 1 - \frac{\sum_{n=1}^{\infty} n|b_n|}{1-\sum_{n=2}^{\infty} n|a_n|} \\ &\ge 1 - \frac{\sum_{n=1}^{\infty} \frac{n(n+\alpha)|\lambda n+\lambda-1|}{1-\alpha}|b_n|}{1-\sum_{n=2}^{\infty} \frac{n(n-\alpha)(\lambda n-\lambda+1)}{1-\alpha}|a_n|} \ge 0, \end{aligned}$$

which proves univalence. Note that f is sense preserving in U, for $0 \le \lambda \le \alpha/(1+\alpha)$ or $\lambda \ge 1/(1+\alpha)$. This is because

$$\begin{aligned}
|h'(z)| &\ge 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \\
&> 1 - \sum_{n=2}^{\infty} n|a_n| \\
&\ge 1 - \sum_{n=2}^{\infty} \frac{n(n-\alpha)(\lambda n - \lambda + 1)}{1-\alpha}|a_n| \\
&\ge \sum_{n=1}^{\infty} \frac{n(n+\alpha)|\lambda n + \lambda - 1|}{1-\alpha}|b_n| \\
&> \sum_{n=1}^{\infty} \frac{n(n+\alpha)|\lambda n + \lambda - 1|}{1-\alpha}|b_n||z|^{n-1} \\
&\ge \sum_{n=1}^{\infty} n|b_n||z|^{n-1} \ge |g'(z)|.
\end{aligned}$$

The functions

$$\begin{aligned}
(5) \qquad f(z) &= z + \sum_{n=2}^{\infty} \frac{1-\alpha}{n(n-\alpha)(\lambda n - \lambda + 1)} x_n z^n \\
&\quad + \sum_{n=1}^{\infty} \frac{1-\alpha}{n(n+\alpha)|\lambda n + \lambda - 1|} \overline{y_n z^n},
\end{aligned}$$

where

$$\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1,$$

show that the coefficient bound given by (4) is sharp. The functions of the form (5) are in $SC_H(\lambda, \alpha)$ because

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left(\frac{n(n-\alpha)(\lambda n - \lambda + 1)}{1-\alpha}|a_n| + \frac{n(n+\alpha)|\lambda n + \lambda - 1|}{1-\alpha}|b_n| \right) \\
&= 1 + \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 2.
\end{aligned}$$

□

THEOREM 2.2. *Let $f = h + \bar{g}$ be so that h and g are given by (3). Then $f \in SC_{\overline{H}}(\lambda, \alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{n(n-\alpha)(\lambda n-\lambda+1)}{1-\alpha}|a_n| + \frac{n(n+\alpha)|\lambda n+\lambda-1|}{1-\alpha}|b_n|\right) \le 2, \tag{6}$$

where $a_1 = 1$, $0 \le \alpha < 1$, $0 \le \lambda \le \alpha/(1+\alpha)$ or $\lambda \ge 1/(1+\alpha)$.

Proof. The if part follows from Theorem 2.1 upon noting that $SC_{\overline{H}}(\lambda, \alpha) \subset SC_H(\lambda, \alpha)$. For the only if part, we show that $f \in SC_{\overline{H}}(\lambda, \alpha)$. Then for $z = re^{i\theta}$ in U we obtain

$$\begin{aligned}
&\operatorname{Re}\left\{\frac{\lambda(z^3h'''(z)-\overline{z^3g'''(z)})+(2\lambda+1)z^2h''(z)+(1-4\lambda)\overline{z^2g''(z)}+zh'(z)+(1-2\lambda)\overline{zg'(z)}}{\lambda(z^2h''(z)+\overline{z^2g''(z)})+zh'(z)+(2\lambda-1)\overline{zg'(z)}} - \alpha\right\} \\
&= \operatorname{Re}\left\{\frac{(1-\alpha)z-\sum_{n=2}^{\infty}n(n-\alpha)(\lambda n-\lambda+1)|a_n|z^n-\sum_{n=1}^{\infty}n(n+\alpha)(\lambda n+\lambda-1)|b_n|\bar{z}^n}{z-\sum_{n=2}^{\infty}n(\lambda n-\lambda+1)|a_n|z^n+\sum_{n=1}^{\infty}n(\lambda n+\lambda-1)|b_n|\bar{z}^n}\right\} \\
&\ge \frac{(1-\alpha)-\left\{\sum_{n=2}^{\infty}n(n-\alpha)(\lambda n-\lambda+1)|a_n|r^{n-1}+\sum_{n=1}^{\infty}n(n+\alpha)|\lambda n+\lambda-1||b_n|r^{n-1}\right\}}{1+\sum_{n=2}^{\infty}n(\lambda n-\lambda+1)|a_n|r^{n-1}+\sum_{n=1}^{\infty}n|\lambda n+\lambda-1||b_n|r^{n-1}} > 0.
\end{aligned}$$

The above inequality must hold for all $z \in U$. In particular, letting $z = r \to 1^-$ yields the required condition (6). □

As special cases of Theorem 2.2, we obtain the following two corollaries.

COROLLARY 2.3. *$f = h + \bar{g} \in SC_{\overline{H}}(0, \alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\frac{n(n-\alpha)}{1-\alpha}|a_n| + \frac{n(n+\alpha)}{1-\alpha}|b_n| \le 2.$$

COROLLARY 2.4. *$f = h + \bar{g} \in SC_{\overline{H}}(1, \alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\frac{n^2(n-\alpha)}{1-\alpha}|a_n| + \frac{n^2(n+\alpha)}{1-\alpha}|b_n| \le 2.$$

Next we determine a representation theorem for functions in $SC_{\overline{H}}(\lambda, \alpha)$.

THEOREM 2.5. *$f = h + \bar{g} \in SC_{\overline{H}}(\lambda, \alpha)$ if and only if f can be expressed as*

$$f(z) = \sum_{n=1}^{\infty}(X_n h_n(z) + Y_n g_n(z)),\ z \in U, \tag{7}$$

where

$$h_1(z) = z,\ h_n(z) = z - \frac{1-\alpha}{n(n-\alpha)(\lambda n-\lambda+1)}z^n, \quad (n = 2, 3, \ldots)$$

and

$$g_n(z) = z + \frac{1-\alpha}{n(n+\alpha)|\lambda n + \lambda - 1|}\bar{z}^n, \quad (n = 1, 2, \ldots),$$

$$\sum_{n=1}^{\infty}(X_n + Y_n) = 1, \ X_n \geq 0 \text{ and } Y_n \geq 0.$$

Proof. For functions f of the form (7) we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}(X_n h_n(z) + Y_n g_n(z)) \\ &= \sum_{n=1}^{\infty}(X_n + Y_n)z - \sum_{n=2}^{\infty}\frac{1-\alpha}{n(n-\alpha)(\lambda n - \lambda + 1)}X_n z^n \\ &\quad + \sum_{n=1}^{\infty}\frac{1-\alpha}{n(n+\alpha)|\lambda n + \lambda - 1|}Y_n \bar{z}^n. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=2}^{\infty}\frac{n(n-\alpha)|\lambda n - \lambda + 1|}{1-\alpha}\left(\frac{1-\alpha}{n(n-\alpha)(\lambda n - \lambda + 1)}X_n\right) \\ &+ \sum_{n=1}^{\infty}\frac{n(n+\alpha)|\lambda n + \lambda - 1|}{1-\alpha}\left(\frac{1-\alpha}{n(n+\alpha)|\lambda n + \lambda - 1|}Y_n\right) \\ = &\sum_{n=2}^{\infty}X_n + \sum_{n=1}^{\infty}Y_n = 1 - X_1 \leq 1 \end{aligned}$$

and so $f \in SC_{\overline{H}}(\lambda, \alpha)$.

Conversely, suppose $f \in SC_{\overline{H}}(\lambda, \alpha)$. Letting

$$X_1 = 1 - \sum_{n=2}^{\infty}X_n - \sum_{n=1}^{\infty}Y_n,$$

where

$$X_n = \frac{n(n-\alpha)(\lambda n - \lambda + 1)}{1-\alpha}|a_n|, \quad (n = 2, 3, \ldots)$$

and

$$Y_n = \frac{n(n+\alpha)|\lambda n + \lambda - 1|}{1-\alpha}|b_n|, \quad (n = 1, 2, \ldots).$$

We obtain the require representation, since

$$\begin{aligned}
f(z) &= z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \bar{z}^n \\
&= z - \sum_{n=2}^{\infty} \frac{(1-\alpha)X_n}{n(n-\alpha)(\lambda n - \lambda + 1)} z^n + \sum_{n=1}^{\infty} \frac{(1-\alpha)Y_n}{n(n+\alpha)|\lambda n + \lambda - 1|} \bar{z}^n \\
&= z - \sum_{n=2}^{\infty} (z - h_n(z)) X_n - \sum_{n=1}^{\infty} (z - g_n(z)) Y_n \\
&= \left(1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n\right) z + \sum_{n=2}^{\infty} h_n(z) X_n + \sum_{n=1}^{\infty} g_n(z) Y_n \\
&= \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_n(z)).
\end{aligned}$$

□

Our next theorem is on the distortion bounds for functions in $SC_{\overline{H}}$ (λ, α), which yields a covering result for this family.

THEOREM 2.6. *If $f \in SC_{\overline{H}}(\lambda, \alpha)$ and $|z| = r < 1$, then*

$$\begin{aligned}
|f(z)| &\leq (1 + |b_1|) r \\
&\quad + \left(\frac{1-\alpha}{2(2-\alpha)(\lambda+1)} - \frac{(1+\alpha)|2\lambda - 1|}{2(2-\alpha)(\lambda+1)} |b_1| \right) r^2, \quad |z| = r < 1
\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &\geq (1 - |b_1|) r \\
&\quad - \left(\frac{1-\alpha}{2(2-\alpha)(\lambda+1)} - \frac{(1+\alpha)|2\lambda - 1|}{2(2-\alpha)(\lambda+1)} |b_1| \right) r^2, \quad |z| = r < 1.
\end{aligned}$$

Proof. We only prove the left hand inequality. The right hand inequality can be proved using similar arguments. Let $f \in SC_{\overline{H}}(\lambda, \alpha)$, then by Theorem 2.2, we obtain

$$\begin{aligned}
&|f(z)| \\
&= \left| z + |b_1| \bar{z} - \sum_{n=2}^{\infty} (|a_n| z^n - |b_n| \bar{z}^n) \right|
\end{aligned}$$

$$
\begin{aligned}
&\geq (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \\
&\geq (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2 \\
&\geq (1-|b_1|)r - \frac{1-\alpha}{2(2-\alpha)(\lambda+1)}r^2 \\
&\quad \times \left(\sum_{n=1}^{\infty} \frac{n(n-\alpha)(\lambda n-\lambda+1)}{1-\alpha}|a_n| + \frac{n(n+\alpha)|\lambda n+\lambda-1|}{1-\alpha}|b_n|\right) \\
&\geq (1-|b_1|)r - \frac{1-\alpha}{2(2-\alpha)(\lambda+1)}\left(1-\frac{(1+\alpha)|2\lambda-1|}{1-\alpha}|b_1|\right)r^2.
\end{aligned}
$$

□

The following covering result follows from the left hand inequality in Theorem 2.7.

COROLLARY 2.7. *If $f \in SC_{\overline{H}}(\lambda,\alpha)$, then*

$$\left\{w : |w| < \frac{(2\lambda+1)(2-\alpha)+1+\{6(\lambda+1)-(1+\alpha)[2(\lambda+1)+|2\lambda-1|]\}|b_1|}{2(2-\alpha)(\lambda+1)}\right\} \subset f(U).$$

For

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \bar{z}^n \text{ and } F(z) = z - \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n \bar{z}^n,$$

we define the convolution of two harmonic functions f and F as

$$(f * F)(z) = f(z) * F(z) = z - \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} b_n B_n \bar{z}^n.$$

Using this definition, we show that the class $SC_{\overline{H}}(\lambda,\alpha)$ is closed under convolution

THEOREM 2.8. *For $0 \leq \beta \leq \alpha < 1$, let $f \in SC_{\overline{H}}(\lambda,\alpha)$ and $F \in SC_{\overline{H}}(\lambda,\beta)$. Then $f * F \in SC_{\overline{H}}(\lambda,\alpha) \subset SC_{\overline{H}}(\lambda,\beta)$.*

Proof. Let $f \in SC_{\overline{H}}(\lambda,\alpha)$ and $F \in SC_{\overline{H}}(\lambda,\beta)$. Obviously, the coefficients of f and F must satisfy conditions similar to the inequality (6).

So for the coefficients of $f * F$ we can write

$$\sum_{n=1}^{\infty} \frac{n(n-\alpha)(\lambda n-\lambda+1)}{1-\alpha}|a_n A_n| + \frac{n(n+\alpha)|\lambda n+\lambda-1|}{1-\alpha}|b_n B_n|$$
$$\leq \sum_{n=1}^{\infty} \frac{n(n-\alpha)(\lambda n-\lambda+1)}{1-\alpha}|a_n| + \frac{n(n+\alpha)|\lambda n+\lambda-1|}{1-\alpha}|b_n|.$$

The right hand side of the above inequality is bounded by 2 because $f \in SC_{\overline{H}}(\lambda, \alpha)$. By the same token, we then conclude that $f * F \in SC_{\overline{H}}(\lambda, \alpha) \subset SC_{\overline{H}}(\lambda, \beta)$. □

THEOREM 2.9. *The class $SC_{\overline{H}}(\lambda, \alpha)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \ldots$ let $f_i \in SC_{\overline{H}}(\lambda, \alpha)$, where f_i is given by

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{i_n}| z^n + \sum_{n=1}^{\infty} |b_{i_n}| \bar{z}^n.$$

Then by (6),

$$\sum_{n=1}^{\infty} \frac{n(n-\alpha)(\lambda n-\lambda+1)}{1-\alpha}|a_n| + \frac{n(n+\alpha)|\lambda n+\lambda-1|}{1-\alpha}|b_n| \leq 2. \tag{8}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{i_n}| \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{i_n}| \right) \bar{z}^n.$$

Then by (8),

$$\sum_{n=1}^{\infty} \left[\frac{n(n-\alpha)(\lambda n-\lambda+1)}{1-\alpha} \left| \sum_{i=1}^{\infty} t_i |a_{i_n}| \right| + \frac{n(n+\alpha)|\lambda n+\lambda-1|}{1-\alpha} \left| \sum_{i=1}^{\infty} t_i |b_{i_n}| \right| \right]$$
$$= \sum_{i=1}^{\infty} t_i \left\{ \sum_{n=1}^{\infty} \left[\frac{n(n-\alpha)(\lambda n-\lambda+1)}{1-\alpha}|a_{i_n}| + \frac{n(n+\alpha)|\lambda n+\lambda-1|}{1-\alpha}|b_{i_n}| \right] \right\}$$
$$\leq 2 \sum_{i=1}^{\infty} t_i = 2.$$

This is the condition by (6) and so

$$\sum_{i=1}^{\infty} t_i f_i(z) \in TS_H^*(\lambda, \alpha).$$

Following [1, 6] we define the $\delta-$neighborhood of a function $f \in S_H$ by

$$N_\delta(f) = \left\{ F \in S_H : F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n z^n} \right.$$
$$\left. \text{and } \sum_{n=2}^{\infty} n(|a_n - A_n| + |b_n - B_n|) + |b_1 - B_1| \leq \delta \right\}.$$

In particular, for the identity function $I(z) = z$, we immediately have

$$\text{(9)} \qquad N_\delta(I) = \left\{ f : f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \right.$$
$$\left. + \sum_{n=1}^{\infty} |b_n| \overline{z}^n \text{and } \sum_{n=2}^{\infty} n(|a_n| + |b_n|) + |b_1| \leq \delta \right\}.$$

□

THEOREM 2.10. *Let*

$$\delta = \frac{1}{(2-\alpha)(\lambda+1)} \left[1 - \alpha + [3(\lambda+1) - (1+\alpha)(\lambda+1+|2\lambda-1|]|b_1|\right].$$

Then

$$SC_{\overline{H}}(\lambda, \alpha) \subset N_\delta(I).$$

Proof. Let $f \in SC_{\overline{H}}(\lambda, \alpha)$. Then the proof follows since, by (6), we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(|a_n| + |b_n|) + |b_1| \\
&\leq |b_1| + \frac{1-\alpha}{(2-\alpha)(\lambda+1)} \sum_{n=2}^{\infty} \left(\frac{n(n-\alpha)(\lambda n - \lambda + 1)}{1-\alpha} |a_n| \right. \\
&\quad \left. + \frac{n(n+\alpha)|\lambda n + \lambda - 1|}{1-\alpha} |b_n| \right) \\
&\leq |b_1| + \frac{1-\alpha}{(2-\alpha)(\lambda+1)} \left[1 - \frac{(1+\alpha)|2\lambda - 1|}{1-\alpha} |b_1| \right] \\
&\leq \frac{1-\alpha}{(2-\alpha)(\lambda+1)} + \frac{3(\lambda+1) - (1+\alpha)(\lambda+1+|2\lambda-1|)}{(2-\alpha)(\lambda+1)} |b_1| = \delta,
\end{aligned}$$

which, in view of definition (9), proves Theorem 2.10. □

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