THE CRB OF SNR ESTIMATES FOR BPSK SIGNALS IN RAYLEIGH FADING CHANNELS WITH CORRELATED DUAL MRC DIVERSITY

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Abstract: The Cramér-Rao bound is derived for SNR estimation of BPSK modulated signals in Rayleigh fading channels with correlated dual MRC diversity.

Keywords: SNR estimation, Cramér-Rao bound, Diversity, Rayleigh fading.

Rayleigh Sönümlü Kanallarda İki Anten Kullanarak En Yüksek Oran Birleştirmeye Dayalı İlintili Çeşitleme Durumunda BPSK Modülasyonlu İşaretlerin İşaret Gürültü Oranı Kestiricilerine İlişkin Cramér-Rao Sınırı

Özet: Rayleigh sönümlü kanallarda iki anten kullanarak en yüksek oran birleştirmeye dayalı ilintili çeşitleme durumunda ikili evre kaydırmalı anahtarlama modülasyonlu işaretlerin işaret gürültü oranı kestiricilerine ilişkin Cramér-Rao sınırı elde edilmiştir.

Anahtar Kelimeler: İşaret gürültü oranı kestirimi, Cramér-Rao sınırı, Çeşitleme, Rayleigh sönümlenme.

1. INTRODUCTION

In many communication systems, where examples include channel quality estimation for link adaptation (Balachandran, Kadaba and Nanda, 1999), optimum decoding of turbo codes on fading channels (Hall and Wilson, 1998), and selection diversity combining (Proakis, 1983), it is necessary to obtain an estimate of the channel signal-to-noise ratio (SNR) from received signals. It is therefore of great interest to assess the performance of any practical SNR estimator in terms of its bias and variance. A well-known lower bound on the variance of any unbiased estimator is the Cramér-Rao bound (CRB) (Kay, 1993), which is particularly useful in making performance predictions of systems before any estimator is actually designed, and can serve as a benchmark for practical estimators. Recently, a CRB work on the SNR estimation of binary phase shift keying (BPSK) signals transmitted over Rayleigh fading channels employing dual maximal ratio combining (MRC) diversity has appeared in Ertaş and Dilaveroğlu (2004), where the correlated branch assumptions were treated very concisely for both data aided (DA) and non-data aided (NDA) cases due to the limitation of space and results were presented only in the form of a plot. However, as it is not virtually possible to obtain uncorrelated fading even with dual branches considering the size limitations of portable hand-held devices with today's technology and current systems, correlated branch fading becomes a more appropriate assumption in the analysis. In this paper, we therefore present a thorough treatment of the correlated fading case in Ertaş and Dilaveroğlu (2004) to fill the gap between the relevant results appearing only as plots and the data model therein.

2. DATA MODEL

As given in Ertaş and Dilaveroğlu (2004), the data model assumes a BPSK signal received by two antennas through flat and slowly Rayleigh fading paths such that the fading remains constant during at

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least one-bit interval. The kth symbol received at the maximal ratio combiner output after coherent demodulation can be written as

$$v_k = \sum_{i=1}^{2} b_k \sqrt{\gamma}(\alpha_i^{(k)})^2 + \alpha_i^{(k)} n_i^{(k)}, \; k = 1,2,\ldots,K, \tag{1}$$

where $b_k = \pm 1$ is the transmitted information bits with equal a priori probability, $\gamma = E_b/N_0$ in which E_b corresponds to the energy per information bit and N_0 is the single-sided power spectral density of the additive white Gaussian noise at the receiver input, $\alpha_i^{(k)}$ and $n_i^{(k)}$ denote the fading amplitude and zero-mean Gaussian noise with variance $1/2$ at the kth bit interval on the ith branch, $i = 1,2,$ respectively, and K is the total number of observed data bits. Assuming sufficient channel interleaving, the fading amplitude $\alpha_i^{(k)}$'s are considered independent for a given branch, and identically Rayleigh distributed and independent of the noise with $E\{(\alpha_i^{(k)})^2\} = 1$ for convenience for any i and k. The joint probability density function (PDF) of $\alpha_1^{(k)}$ and $\alpha_2^{(k)}$ is given by

$$p_{\alpha_1^{(k)},\alpha_2^{(k)}}(r_1,r_2) = \frac{4r_1r_2}{1-\rho}\exp\left[-\frac{1}{1-\rho}(r_1^2+r_2^2)\right] I_0\left(\frac{2\sqrt{\rho}\,r_1r_2}{1-\rho}\right), \; r_1,r_2 \geq 0, \; 0 \leq \rho < 1, \tag{2}$$

where $\rho = \mathrm{cov}(r_1^2, r_2^2)\big/\sqrt{\mathrm{var}(r_1^2)\,\mathrm{var}(r_2^2)}$ is the power correlation coefficient between the two branches and $I_0(\cdot)$ is the modified Bessel function of the first kind with order zero. The average SNR at the combiner output is $\dfrac{E_b}{2N_0}\sum_{i=1}^{2} E\{(\alpha_i^{(k)})^2\} = \dfrac{E_b}{N_0}$, and our interest is to find the CRB on the variance of any unbiased estimator of the SNR $\gamma = \dfrac{E_b}{N_0}$, which we denote by CRB_γ, by using the observed data $\{v_k\}_{k=1}^{K}$.

3. THE CRB FOR NDA ESTIMATION

Treating the correlation coefficient ρ as an unknown parameter, the CRB_γ is given by (Kay, 1993)

$$\mathrm{CRB}_\gamma = [I^{-1}(\gamma,\rho)]_{11}, \tag{3}$$

where $I(\gamma,\rho)$ is the Fisher information matrix (FIM) of (γ,ρ) given by

$$I(\gamma,\rho) = K \cdot \begin{bmatrix} -E_{v_k}\left\{\dfrac{\partial^2 \ln p(v_k;\gamma,\rho)}{\partial\gamma^2}\right\} & -E_{v_k}\left\{\dfrac{\partial^2 \ln p(v_k;\gamma,\rho)}{\partial\gamma\partial\rho}\right\} \\ -E_{v_k}\left\{\dfrac{\partial^2 \ln p(v_k;\gamma,\rho)}{\partial\rho\partial\gamma}\right\} & -E_{v_k}\left\{\dfrac{\partial^2 \ln p(v_k;\gamma,\rho)}{\partial\rho^2}\right\} \end{bmatrix}, \tag{4}$$

in which $\ln p(v_k;\gamma,\rho)$ is the log-likelihood function of (γ,ρ) for the kth data sample v_k, $E_{v_k}\{\cdot\}$ denotes expectation with respect to v_k, and $[I^{-1}(\gamma,\rho)]_{11}$ is the first row-first column element of $I^{-1}(\gamma,\rho)$. By using (1) and (2) we obtain the likelihood function of (γ,ρ) for the non-data-aided estimation as

$$p(v_k;\gamma,\rho) = \frac{1}{2\sqrt{\rho}}\left(\frac{e^{-2A|v_k|}}{A} - \frac{e^{-2B|v_k|}}{B}\right)\cosh(2\sqrt{\gamma}v_k), \tag{5}$$

where $A=\sqrt{\gamma+\frac{1}{1+\sqrt{\rho}}}$ and $B=\sqrt{\gamma+\frac{1}{1-\sqrt{\rho}}}$.

Using (5), we obtain the elements of the FIM in (4) as follows:

$$
\begin{aligned}
E_{v_k}\left\{\frac{\partial^2 \ln p(v_k;\gamma,\rho)}{\partial\gamma^2}\right\} &= a_{\gamma\gamma,1}b_1 f_4(A)+a_{\gamma\gamma,1}b_2 f_4(B)+a_{\gamma\gamma,2}b_1 f_3(A)+a_{\gamma\gamma,2}b_2 f_3(B)\\
&-\frac{a_{\gamma,1}^2 b_1}{a_1}f_5(A)-\frac{a_{\gamma,1}^2 b_2}{a_1}f_5(B)\\
&+a_1 b_{\gamma\gamma,1}f_1(A)+a_1 b_{\gamma\gamma,2}f_2(A)+a_1 b_{\gamma\gamma,3}f_3(A)\\
&+a_1 b_{\gamma\gamma,4}f_1(B)+a_1 b_{\gamma\gamma,5}f_2(B)+a_1 b_{\gamma\gamma,6}f_3(B)\\
&-a_1 b_{\gamma,1}^2 f_6(A)-a_1 b_{\gamma,2}^2 f_8(A)-a_1 b_{\gamma,3}^2 f_6(B)-a_1 b_{\gamma,4}^2 f_8(B)\\
&-2a_1 b_{\gamma,1}b_{\gamma,2}f_7(A)-2a_1 b_{\gamma,1}b_{\gamma,3}f_6(\frac{A+B}{2})\\
&-2a_1(b_{\gamma,1}b_{\gamma,4}+b_{\gamma,2}b_{\gamma,3})f_7(\frac{A+B}{2})\\
&-2a_1 b_{\gamma,2}b_{\gamma,4}f_8(\frac{A+B}{2})-2a_1 b_{\gamma,3}b_{\gamma,4}f_7(B),
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
E_{v_k}\left\{\frac{\partial^2 \ln p(v_k;\gamma,\rho)}{\partial\rho^2}\right\} &= a_{\rho\rho,1}b_1 f_1(A)+a_{\rho\rho,1}b_2 f_1(B)\\
&-\frac{a_{\rho,1}^2 b_1}{a_1}f_1(A)-\frac{a_{\rho,1}^2 b_2}{a_1}f_1(B)\\
&+a_1 b_{\rho\rho,1}f_1(A)+a_1 b_{\rho\rho,2}f_2(A)+a_1 b_{\rho\rho,3}f_3(A)\\
&+a_1 b_{\rho\rho,4}f_1(B)+a_1 b_{\rho\rho,5}f_2(B)+a_1 b_{\rho\rho,6}f_3(B)\\
&-a_1 b_{\rho,1}^2 f_6(A)-a_1 b_{\rho,2}^2 f_8(A)-a_1 b_{\rho,3}^2 f_6(B)-a_1 b_{\rho,4}^2 f_8(B)\\
&-2a_1 b_{\rho,1}b_{\rho,2}f_7(A)-2a_1 b_{\rho,1}b_{\rho,3}f_6(\frac{A+B}{2})\\
&-2a_1(b_{\rho,1}b_{\rho,4}+b_{\rho,2}b_{\rho,3})f_7(\frac{A+B}{2})\\
&-2a_1 b_{\rho,2}b_{\rho,4}f_8(\frac{A+B}{2})-2a_1 b_{\rho,3}b_{\rho,4}f_7(B),
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
E_{v_k}\left\{\frac{\partial^2 \ln p(v_k;\gamma,\rho)}{\partial\rho\partial\gamma}\right\} &= a_{\rho\gamma,1}b_1 f_4(A)+a_{\rho\gamma,1}b_2 f_4(B)\\
&-\frac{a_{\gamma,1}a_{\rho,1}b_1}{a_1}f_4(A)-\frac{a_{\gamma,1}a_{\rho,1}b_2}{a_1}f_4(B)\\
&+a_1 b_{\rho\gamma,1}f_1(A)+a_1 b_{\rho\gamma,2}f_2(A)+a_1 b_{\rho\gamma,3}f_3(A)\\
&+a_1 b_{\rho\gamma,4}f_1(B)+a_1 b_{\rho\gamma,5}f_2(B)+a_1 b_{\rho\gamma,6}f_3(B)\\
&-a_1 b_{\gamma,1}b_{\rho,1}f_6(A)-a_1(b_{\gamma,1}b_{\rho,2}+b_{\gamma,2}b_{\rho,1})f_7(A)\\
&-a_1(b_{\gamma,1}b_{\rho,3}+b_{\gamma,3}b_{\rho,1})f_6(\frac{A+B}{2})\\
&-a_1(b_{\gamma,1}b_{\rho,4}+b_{\gamma,4}b_{\rho,1}+b_{\gamma,2}b_{\rho,3}+b_{\gamma,3}b_{\rho,2})f_7(\frac{A+B}{2})\\
&-a_1 b_{\gamma,2}b_{\rho,2}f_8(A)-a_1(b_{\gamma,2}b_{\rho,4}+b_{\gamma,4}b_{\rho,2})f_8(\frac{A+B}{2})\\
&-a_1 b_{\gamma,3}b_{\rho,3}f_6(A)-a_1(b_{\gamma,3}b_{\rho,4}+b_{\gamma,4}b_{\rho,3})f_7(B)\\
&-a_1 b_{\gamma,4}b_{\rho,4}f_8(B),
\end{aligned}
\tag{8}
$$

where in (6)-(8)

$$a_1=\frac{1}{2\sqrt{\rho}},\ a_{\gamma,1}=\frac{1}{2\sqrt{\rho}\sqrt{\gamma}},\ a_{\gamma\gamma,1}=-\frac{1}{4\sqrt{\rho}\gamma^{3/2}},\ a_{\gamma\gamma,2}=\frac{1}{2\sqrt{\rho\gamma}},\ a_{\rho,1}=-\frac{1}{4\rho^{3/2}},\ a_{\rho\rho,1}=\frac{3}{8\rho^{5/2}},$$

$$a_{\rho\gamma,1}=-\frac{1}{4\rho^{3/2}\sqrt{\gamma}},$$

$$b_1=\frac{1}{A},\ b_2=-\frac{1}{B},\ b_{\gamma,1}=-\frac{A_\gamma}{A^2},\ b_{\gamma,2}=-\frac{2A_\gamma}{A},\ b_{\gamma,3}=-\frac{B_\gamma}{B^2},\ b_{\gamma,4}=\frac{2B_\gamma}{B},\ b_{\gamma\gamma,1}=\frac{2A_\gamma^2}{A^3}-\frac{A_{\gamma\gamma}}{A^2},$$

$$b_{\gamma\gamma,2}=\frac{4A_\gamma^2}{A^2}-\frac{2A_{\gamma\gamma}}{A},\ b_{\gamma\gamma,3}=\frac{4A_\gamma^2}{A},\ b_{\gamma\gamma,4}=-\frac{2B_\gamma^2}{B^3}+\frac{B_{\gamma\gamma}}{B^2},\ b_{\gamma\gamma,5}=-\frac{4B_\gamma^2}{B^2}+\frac{2B_{\gamma\gamma}}{B},\ b_{\gamma\gamma,6}=-\frac{4B_\gamma^2}{B},$$

$$b_{\rho,1}=-\frac{A_\rho}{A^2},\ b_{\rho,2}=-\frac{2A_\rho}{A},\ b_{\rho,3}=\frac{B_\rho}{B^2},\ b_{\rho,4}=\frac{2B_\rho}{B},\ b_{\rho\rho,1}=\frac{2A_\rho^2}{A^3}-\frac{A_{\rho\rho}}{A^2},\ b_{\rho\rho,2}=\frac{4A_\rho^2}{A^2}-\frac{2A_{\rho\rho}}{A},$$

$$b_{\rho\rho,3}=\frac{4A_\rho^2}{A},\ b_{\rho\rho,4}=-\frac{2B_\rho^2}{B^3}+\frac{B_{\rho\rho}}{B^2},\ b_{\rho\rho,5}=-\frac{4B_\rho^2}{B^2}+\frac{2B_{\rho\rho}}{B},\ b_{\rho\rho,6}=-\frac{4B_\rho^2}{B},$$

$$b_{\rho\gamma,1}=\frac{2A_\rho A_\gamma}{A^3}-\frac{A_{\rho\gamma}}{A^2},\ b_{\rho\gamma,2}=\frac{4A_\rho A_\gamma}{A^2}-\frac{2A_{\rho\gamma}}{A},\ b_{\rho\gamma,3}=\frac{4A_\rho A_\gamma}{A},\ b_{\rho\gamma,4}=-\frac{2B_\rho B_\gamma}{B^3}+\frac{B_{\rho\gamma}}{B^2},$$

$$b_{\rho\gamma,5}=-\frac{4B_\rho B_\gamma}{B^2}+\frac{2B_{\rho\gamma}}{B},\ b_{\rho\gamma,6}=-\frac{4B_\rho B_\gamma}{B},$$

$$f_1(r)=\frac{r}{r^2-\gamma},\ f_2(r)=\frac{r^2+\gamma}{2(r^2-\gamma)^2},\ f_3(r)=\frac{r(r^2+3\gamma)}{2(r^2-\gamma)^3},\ f_4(r)=\frac{r\sqrt{\gamma}}{(r^2-\gamma)^2},$$

$$f_5(r)=\frac{1}{256\gamma^{3/2}}\left[\zeta\left(3,\frac{1}{4}\left(-1+\frac{r}{\sqrt{\gamma}}\right)\right)-3\zeta\left(3,\frac{1}{4}\left(1+\frac{r}{\sqrt{\gamma}}\right)\right)+3\zeta\left(3,\frac{1}{4}\left(3+\frac{r}{\sqrt{\gamma}}\right)\right)-\zeta\left(3,\frac{1}{4}\left(5+\frac{r}{\sqrt{\gamma}}\right)\right)\right],$$

$$f_6(r)=\int_{-\infty}^{\infty}\frac{e^{-4r|v|}\cosh(2\sqrt{\gamma}v)}{b_1e^{-2A|v|}+b_2e^{-2B|v|}}dv,\ f_7(r)=\int_{-\infty}^{\infty}\frac{|v|e^{-4r|v|}\cosh(2\sqrt{\gamma}v)}{b_1e^{-2A|v|}+b_2e^{-2B|v|}}dv,$$

$$f_8(r)=\int_{-\infty}^{\infty}\frac{v^2e^{-4r|v|}\cosh(2\sqrt{\gamma}v)}{b_1e^{-2A|v|}+b_2e^{-2B|v|}}dv.$$

Here, in the equalities $f_1(r)$ to $f_5(r)$, r takes the values A and B in turn, and A, B, and $\frac{A+B}{2}$ in the equalities $f_6(r)$ to $f_8(r)$ involving numerical integrations. $\zeta(s,a)=\sum_{n=0}^{\infty}(n+a)^{-s}$ is the generalized Riemann's zeta function. The CRB can then be obtained by substituting (6)-(8) into (4) and then finally into (3).

4. THE CRB FOR DA ESTIMATION

For the data-aided estimation, removing the dependency of v_k on the information bits b_k in (1), we obtain the likelihood function of (γ,ρ) for the kth data sample v_k as

$$p(v_k;\gamma,\rho) = \frac{1}{2\sqrt{\rho}}\left(\frac{e^{-2A|v_k|}}{A} - \frac{e^{-2B|v_k|}}{B}\right)\exp(2\sqrt{\gamma}v_k)\,.$$

Using this PDF in (4), we find surprisingly that the elements of the FIM for the DA estimation appear to be exactly the same as those for the NDA estimation except for the case in which only $f_5(A)$ and $f_5(B)$ should read as $f_3(A)$ and $f_3(B)$, respectively, in the first row-first column element of (4) in (6). Now, obtaining the CRB is then straightforward in the same way as in Section 3.

5. CONCLUSION

We have derived the Cramér-Rao bound with details on data and non-data-aided SNR estimation of BPSK signals in Rayleigh fading channels employing correlated dual MRC diversity. The derived expressions constructing the bound, which were not included in Ertaş and Dilaveroğlu (2004), can now be directly used to obtain the bound.

6. REFERENCES

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