LOKAL OLMAYAN HİPERBOLİK PROBLEMLER İÇİN KARARLI FARK ŞEMALARI

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ÖZET

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Bilindiği gibi, hiperbolik denklemler için Cauchy ve lokal olmayan sınır değer problemleri, kendine eşlenik pozitif tanımlı operatör A yardımıyla, Hilbert uzayı H da bir adi diferansiyel denklem için Cauchy ve lokal olmayan sınır değer problemlerine indirgenebilir. Operatör yaklaşımı kullanılarak bu indirgenmiş problemlerin çözümleri için kararlılık tahminleri elde edilebilir. Bu soyut sonuç, uygulamalarda, hiperbolik denklemlerde Cauchy probleminin ve lokal olmayan sınır değer probleminin çözümleri için kararlılık kestirimleri elde etmemize müsaade eder. Bu çalışmada, soyut Cauchy ve lokal olmayan sınır değer problemlerinin yaklaşık çözümü için, A nın tam kuvvetleri ile oluşturulan, üçüncü ve dördüncü mertebeden kararlı doğruluk fark şemaları verilmiş ve bu fark şemalarının çözümleri için kararlılık kestirimleri elde edilmiştir. Ayrıca bu fark şemalarının çözümleri için teorik sonuçların doğruluğu nümerik örneklerle desteklenmiştir.

Anahtar Kelimeler: Hiperbolik denklemler, Fark şemaları, Yakınsaklık, Kararlılık, Sayısal analiz

2011, vii+170 sayfa.

ABSTRACT

PhD Thesis

STABLE DIFFERENCE SCHEMES FOR THE NONLOCAL HYPERBOLIC PROBLEMS

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It is known that various Cauchy problem and nonlocal boundary value problem for the hyperbolic equations can be reduced to the Cauchy problem and the nonlocal boundary problem for the differential equation in a Hilbert space H with self-adjoint positive definite operator A. Applying the operator approach we obtain the stability estimates for the solutions of these problems. In applications this abstract result permit us to obtain the stability estimates for solutions of Cauchy problem and nonlocal boundary value problem for hyperbolic equations. The third and fourth order of accuracy difference schemes generated by the integer power of A approximately solving this abstract Cauchy and nonlocal boundary value problems are presented. The stability estimates for the solution of these difference schemes are obtained. The theoretical statements for the solution of these difference schemes are supported by the results of numerical experiments.

Key words: Hyperbolic equation, Difference schemes, Convergence, Stability, Numerical analysis
2011, vii+170 pages.

ÖNSÖZ VE TEŞEKKÜR

Öncelikle öğrencilerini çocukları gözüyle gören ve onlardan desteğini ve danışmanlığını sadece akademik alanda değil özel hayatlarındaki sorunlarında da esirgemeyen, akademik hayata atıldığım günden bugüne kadar alanındaki uzmanlığıyla derin ve değerli bilgilerini benimle paylaşmaktan çekinmeyen, bu sayede karşılaşmış olduğum birçok problemde aydınlatıcı bilgileriyle yol gösteren ve çalışmış olduğum alanlarda önümde derin ufuklar açan hocam sayın Prof. Dr. Allaberen ASHYRALYEV' e sonsuz teşekkürlerimi sunarım.

Doktora eğitimim içerisinde pek çok konuda karşılaştığım problemlere değerli görüş ve bilgileriyle çözümler üretme konusunda yardımlarını hiçbir zaman benden esirgemeyen sayın Prof. Dr. Mehmet ÇAĞLIYAN' a da çok teşekkür ederim.

Bu çalışma yeni doğan kızıma ithaf edilmiştir.

İÇİNDEKİLER

Sayfa

ŞEKİLLER DİZİNİ

Sayfa

ÇİZELGELER DİZİNİ

Sayfa

1. GİRİŞ

Son yıllarda lokal olmayan sınır değer problemleri hızla büyüyen bir bilimsel araştırma alanı haline gelmiştir. Bu tip problemler üzerine çalışmalar sadece teorik amaçlı ortaya çıkmamıştır, aynı zamanda mühendislik ve fizikte birçok olayın modellemesinde de kullanılmaktadır. Örneğin kuantum mekaniği, akustik, elektromanyetik, akışkanlar mekaniğinde çoğu problemler ve fiziğin diğer alanları için hiperbolik tip kısmi diferansiyel denklemler kullanılmaktadır (Sobolevskii ve Chebotaryeva 1977, Ashyralyev ve Muradov 1995, Ashyralyev ve Muradov 1998, Ashyralyev ve Sobolevskii 2001, Ashyralyev ve Sobolevskii 2004, Ashyralyev ve Aggez 2004, Ashyralyev ve Aggez 2011, Ashyralyev ve Sobolevskii 2005, Ashyralyev ve Koksal 2005, Ashyralyev ve Koksal 2007, Ashyralyev ve Koksal 2008, Ashyralyev ve Koksal 2009, Ashyralyev ve Ozdemir 2005, Ashyralyev ve Ozdemir 2007, Ashyralyev ve Ozdemir 2009, Ashyralyev ve Ozdemir 2010, Ashyralyev ve ark. 2009, Ashyralyev ve Dal 2009, Ashyralyev ve Dal 2011, Ashyralyev ve ark. 2010, Ashyralyev ve ark. 2011, Berdyshev ve Karimov 2006).

Bilindiği üzere çeşitli hiperbolik tip sınır değer problemleri Hilbert uzayı *H'* da kendine eşlenik pozitif tanımlı *A* operatörü yardımı ile

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = f(t), 0 \le t \le 1, \\ u(0) = \varphi, u'(0) = \psi \end{cases} \tag{1.1}$$

başlangıç değer problemine indirgenebilir.

Sobolevskii ve Chebotaryeva 1977 makalesinde, problem (1.1) in yaklaşık çözümü için birinci mertebeden doğruluk fark şeması

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_{k+1} = f_k, \\ f_k = f(t_k),\ t_k = k\tau,\ 1 \le k \le N-1, N\tau = 1, \\ \tau^{-1}(u_1 - u_0) + iA^{1/2}u_1 = iA^{1/2}u_0 + \psi, u_0 = \varphi \end{cases} \tag{1.2}$$

sunulmuştur. Fark şeması (1.2) nin çözümü için kararlılık kestirimleri elde edilmiştir. Ashyralyev ve Muradov 1995, Ashyralyev ve Muradov 1998 makalelerinde Cauchy problemi (1.1) in yaklaşık çözümünde ikinci mertebeden doğruluk fark şemaları

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_k+\frac{\tau^2}{4}A^2u_{k+1}=f_k, \\ f_k=f(t_k), 1\le k\le N-1, N\tau=1, \\ \tau^{-1}(u_1-u_0)+iA^{1/2}(I+\frac{i\tau}{2}A^{1/2})u_1=z_1, \\ z_1=(I+i\tau A^{1/2})\psi+\frac{\tau}{2}f_0+(iA^{\frac{1}{2}}-\tau A)u_0, \\ u_0=\varphi, f_0=f(0), \end{cases} \tag{1.3}$$

ve

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{1}{4}A(u_{k+1}+2u_k+u_{k-1})=f_k, \\ f_k=f(t_k), 1\le k\le N-1, N\tau=1, \\ \tau^{-1}(u_1-u_0)+\frac{i}{2}A^{1/2}(u_1+u_0)=z_1, \\ z_1=(I+\frac{i\tau}{2}A^{1/2})\psi+\frac{\tau}{2}f_0+(iA^{\frac{1}{2}}-\frac{\tau A}{2})u_0, \\ u_0=\varphi, f_0=f(0) \end{cases} \tag{1.4}$$

için benzer sonuçları elde etmiştir. Buna rağmen, bu fark şemalarının pratik gerçekleştirilmeleri için bir $A^{1/2}$ operatörü inşa etmek gereklidir. Bu işlem bilgisayar için çok zordur. Bu sebeple, teorik sonuçlara rağmen, başlangıç değer probleminin nümerik çözümüne uygulaması iyi değildir. Ashyralyev ve Sobolevskii 2001 makalelerinde, A operatörünün tamsayı kuvvetlerini kullanarak başlangıç değer problemi (1.1) in yaklaşık çözümü için birinci ve ikinci mertebeden doğruluk fark şemalarını ve bu fark şemalarının çözümleri için kararlılık kestirimlerini elde etmiştir.

Ashyralyev ve Aggez 2004 makalesinde, A operatörünün tamsayı kuvvetlerini kullanarak, iki noktada lokal olmayan hiperbolik sınır problemi

$$\begin{cases} \frac{d^2u(t)}{dt^2}+Au(t)=f(t), 0\le t\le 1, \\ u(0)=\alpha u(1)+\varphi, \\ u'(0)=\beta u'(1)+\psi \end{cases} \tag{1.5}$$

in yaklaşık çözümü için birinci ve ikinci mertebeden kararlı fark şemaları sunulmuştur. Problem (1.5) in ve fark şemalarının çözümü için kararlılık kestirimleri elde edilmiştir Uygulamada da iki noktada lokal olmayan hiperbolik tip sınır değer problemi için kararlılık kestirimleri elde edilmiştir.

Ashyralyev ve Yildirim 2010 makalesinde, A operatörünün tamsayı kuvvetlerini kullanarak çok noktada lokal olmayan hiperbolik sınır problemi

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = f(t), 0 \le t \le 1, \\ u(0) = \sum_{r=1}^{n} \alpha_r u(\lambda_r) + \varphi, \\ u_t(0) = \sum_{r=1}^{n} \beta_r u_t(\lambda_r) + \psi \end{cases} \tag{1.6}$$

nın yaklaşık çözümü için birinci ve ikinci mertebeden kararlı fark şemaları sunulmuş ve fark şemalarının çözümü için kararlılık kestirimleri elde edilmiştir. Uygulamada da çok noktada lokal olmayan hiperbolik tip sınır değer problemi için kararlılık kestirimleri elde edilmiştir.

Parabolik, eliptik ve karışık tip lokal olmayan sınır değer problemleri için iyi tanımlanmışlık üzerine birçok araştırma yapılmıştır (Krein 1966, Salahatdinov 1974, Sobolevskii 1975, Dzhuraev 1979, Fattorini 1985, Salakhitdinov ve Urinov 1997, Piskarev ve Shaw 1997, Gulin ve ark. 2001, Ashyralyev 2006, Gordezani ve ark. 2003, Agarwal ve ark. 2005 , Somali ve Oger 2005).

Bu tez çalışmasında, hiperbolik tip Cauchy problemi (1.1) ve buradan elde edilecek sonuçlardan faydalanılarak lokal olmayan sınır değer problemi (1.5) in yaklaşık çözümleri için üçüncü ve dördüncü mertebeden kararlı doğruluk fark şemaları ve bunların nümerik çözümleri için kararlılık kestirimleri elde edilmiş; nümerik yöntemlerle, çözümleri, analizleri ve uygulamalarının geliştirilmesine çalışılmıştır. Tezin bölümleri ve içerikleri aşağıda kısaca özetlenmektedir.

Tez beş ana bölümden oluşmaktadır. Birinci bölüm giriş bölümüdür. İkinci bölüm üç alt bölümü ihtiva etmektedir. Bunlardan birincisi problemin sunulması ve formüllerin elde edilmesi konularını içermektedir. İkincisi hiperbolik tip Cauchy problemi (1.1) in yaklaşık çözümü için A operatörünün tam kuvvetleri kullanılarak elde edilen üçüncü mertebeden kararlı doğruluk fark şemalarının elde edilmesi konusunu içermektedir. Üçüncüsü hiperbolik tip Cauchy problemi (1.1) in yaklaşık çözümü için A operatörünün tam kuvvetleri kullanılarak elde edilen dördüncü mertebeden kararlı doğruluk fark şemalarının elde edilmesi hakkındadır. Üçüncü bölüm üç alt bölümü ihtiva etmektedir. Bunlardan birincisi problemin sunulması hususundadır. İkincisi hiperbolik tip lokal olmayan sınır değer problemi (1.5) in yaklaşık çözümü için A operatörünün tam kuvvetleri kullanılarak elde edilen üçüncü mertebeden kararlı doğruluk fark şemalarının elde edilmesi hakkındadır. Üçüncüsü hiperbolik tip lokal olmayan sınır değer problemi (1.5) in yaklaşık çözümü için A operatörünün tam kuvvetleri kullanılarak elde edilen dördüncü mertebeden kararlı doğruluk fark şemalarının elde edilmesi hakkındadır. Dördüncü bölüm nümerik sonuçları ihtiva etmektedir. Hiperbolik tip Cauchy problemi (1.1) in ve lokal olmayan sınır değer problemi (1.5) in yaklaşık çözümü için A operatörünün tam kuvvetleri kullanılarak elde edilen üçüncü ve dördüncü mertebeden kararlı doğruluk fark şemalarının nümerik uygulamaları ele alınmış, bu fark şemalarının Modifiye Edilmiş Gauss Eliminasyon metodu kullanılarak matrislerden oluşan bir lineer sisteme dönüştürülmesi ele alınmıştır. Ayrıca problemlerin matris sistemindeki çözümleri için, Matlab programı kullanılarak elde edilen grafikler ve hata analizlerini ihtiva eden bir tablo sunulmuştur. Beşinci bölüm sonuçlardan müteşekkildir.

2. HİPERBOLİK TİP DİFERANSİYEL DENKLEMLERDE CAUCHY PROBLEMİ İÇİN KARARLI FARK ŞEMALARI

2.1 Hiperbolik Tip Cauchy Problemi

Bu bölümde Hilbert uzayı H da kendine eşlenik, pozitif tanımlı A operatörü yardımı ile oluşturulan genel formda hiperbolik denklemler için Cauchy problemi

$$u''(t) + Au(t) = f(t), 0 < t < T, u(0) = \varphi, u'(0) = \psi \tag{2.1}$$

ele alındı. Bir $u(t)$ fonksiyonu için aşağıdaki şartlar sağlanıyorsa $u(t)$ ye problem (2.1) in çözümü denir;

i) $u(t)$, $[0,1]$ aralığında iki kez türevlenebilir ve süreklidir. Kapalı aralığın sınır noktalarındaki türevler bir taraflı olarak mevcuttur,

ii) $u(t)$, her $t \in [0,1]$ için, $D(A)$ tanım kümesinin elemanıdır ve $Au(t)$ fonksiyonu $[0,1]$ aralığında süreklidir,

iii) $u(t)$, (2.1) denklemini ve başlangıç şartlarını sağlar.

Eğer bir $f(t)$ fonksiyonu $[0,T]$ kapalı aralığında sadece sürekli değil aynı zamanda da türevlenebilir ve $\varphi \in D(A)$, $\psi \in D(A^{\frac{1}{2}})$ ise formül

$$u(t) = c(t)\varphi + s(t)\psi + \int_0^t s(t-\lambda) f(\lambda) d\lambda, \tag{2.2}$$

problem (2.1) in bir çözümüdür (Fattorini 1985). Operatör teoride, tanım aralığında sürekli $\{c(t), t \geq 0\}$ kosinüs operatör fonksiyonu,

$$c(t) = \frac{e^{itA^{1/2}} + e^{-itA^{1/2}}}{2}$$

formülü ile tanımlanır. Sinüs operatör fonksiyonu için de

$$s(t)u = \int_0^t c(s)u \, ds$$

tanımından yararlanarak

$$s(t) = A^{-1/2} \frac{e^{itA^{1/2}} - e^{-itA^{1/2}}}{2i}$$

formülü yazılabilir. Kosinüs operatör fonksiyonu teorisi için Fattorini 1985, Piskarev ve Shaw 1997 çalışmalarından yararlanıldı.

$[0,T]$ kapalı aralığında τ adımlı düzgün bir grid (ızgara)

$$[0,T]_\tau = \{t_k = k\tau, k = 0,1,\ldots,N, N\tau = T\}$$

ele alınsın. Başlangıç değer problemi (2.1) in çözümünün iki adımlı fark şemalarının oluşturması için, öncelikle diferansiyel operatörün ve $u'(0)$ türevinin yaklaşımlarının elde edilmesi gerekir.

Önce diferansiyel denklem (2.1) in yaklaşımını ele alalım. Taylor ayrışım (dekompozisyon) metodu üç noktada uygulanarak (2.1) denkleminin üçüncü mertebeden yaklaşım formülü

$$\begin{aligned} &u(t_{k+1}) - 2u(t_k) + u(t_{k-1}) - \frac{2}{3}\tau^2 u''(t_k) - \frac{1}{6}\tau^2 \left(u''(t_{k+1}) + u''(t_{k-1})\right) \\ &+ \frac{1}{12}\tau^4 u^{(4)}(t_{k+1}) = o(\tau^5) \end{aligned} \tag{2.3}$$

ve dördüncü mertebeden yaklaşım formülü

$$u\left(t_{k+1}\right)-2u\left(t_{k}\right)+u\left(t_{k-1}\right)-\frac{5}{6}\tau^{2}u^{''}\left(t_{k}\right)-\frac{1}{12}\tau^{2}\left(u^{''}\left(t_{k+1}\right)+u^{''}\left(t_{k-1}\right)\right)$$

$$-\frac{1}{72}\tau^{4}u^{(4)}\left(t_{k}\right)+\frac{1}{144}\tau^{4}\left(u^{(4)}\left(t_{k+1}\right)+u^{(4)}\left(t_{k-1}\right)\right)=o\left(\tau^{6}\right) \qquad (2.4)$$

elde edilir (Ashyralyev ve Sobolevskii 2004). Ayrıca problem (2.1) den yararlanarak

$$u''\left(t_{k}\right)=-Au\left(t\right)+f\left(t\right),$$

$$u^{(4)}\left(t\right)=-Au''\left(t\right)+f''\left(t\right)=A^{2}u\left(t\right)-Af\left(t\right)+f''\left(t\right) \qquad (2.5)$$

yazılabilir. (2.3) ve (2.5) denklemleri yardımı ile, problem (2.1) in çözümü için üçüncü mertebeden yaklaşım formülü

$$\frac{u\left(t_{k+1}\right)-2u\left(t_{k}\right)+u\left(t_{k-1}\right)}{\tau^{2}}-\frac{2}{3}\left(-Au\left(t_{k}\right)+f\left(t_{k}\right)\right)$$

$$-\frac{1}{6}\left(-A\left(u\left(t_{k+1}\right)+u\left(t_{k-1}\right)\right)+f\left(t_{k+1}\right)+f\left(t_{k-1}\right)\right)$$

$$+\frac{1}{12}\tau^{2}\left(A^{2}u\left(t_{k+1}\right)-Af\left(t_{k+1}\right)+f^{''}\left(t_{k+1}\right)\right)=o\left(\tau^{3}\right),$$

elde edilir. Benzer şekilde, denklemler (2.4) ve (2.5) yardımı ile de dördüncü mertebeden yaklaşım formülü

$$\frac{u(t_{k+1})-2u(t_k)+u(t_{k-1})}{\tau^2}-\frac{5}{6}\left(-Au(t_k)+f(t_k)\right)$$

$$-\frac{1}{12}\left(-A\left(u(t_{k+1})+u(t_{k-1})\right)+f(t_{k+1})+f(t_{k-1})\right)$$

$$-\frac{1}{72}\tau^2\left(A^2u(t_k)-Af(t_k)+f^{''}(t_k)\right)+\frac{1}{144}\tau^2\left(A^2\left(u(t_{k+1})+u(t_{k-1})\right)\right.$$

$$\left.-A\left(f(t_{k+1})+f(t_{k-1})\right)+f^{''}(t_{k+1})+f^{''}(t_{k-1})\right)=o\left(\tau^4\right)$$

bulunur. Sondaki küçük terimler ihmal edilerek, problem (2.1) in üçüncü ve dördüncü mertebeden yaklaşım formülleri sırasıyla

$$\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{2}{3}Au_k+\frac{1}{6}A\left(u_{k+1}+u_{k-1}\right)+\frac{1}{12}\tau^2A^2u_{k+1}=f_k,$$

$$f_k=\frac{2}{3}f(t_k)+\frac{1}{6}\left(f(t_{k+1})+f(t_{k-1})\right)-\frac{1}{12}\tau^2\left(-Af(t_{k+1})+f^{''}(t_{k+1})\right) \tag{2.6}$$

ve

$$\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{5}{6}Au_k+\frac{1}{12}A\left(u_{k+1}+u_{k-1}\right)$$

$$-\frac{1}{72}\tau^2A^2u_k+\frac{1}{144}\tau^2A^2\left(u_{k+1}+u_{k-1}\right)=f_k,$$

$$f_k=\frac{5}{6}f(t_k)+\frac{1}{12}\left(f(t_{k+1})+f(t_{k-1})\right)+\frac{1}{72}\tau^2\left(-Af(t_k)+f''(t_k)\right)$$

$$-\frac{1}{144}\tau^2\left(-A\left(f(t_{k+1})+f(t_{k-1})\right)+f''(t_{k+1})+f''(t_{k-1})\right) \tag{2.7}$$

olarak yazılabilir. İkinci olarak, başlangıç değer $u'(0)$ ın yaklaşımını ele alalım. (2.2) formülünden yararlanarak

$$\frac{u(\tau)-u(0)}{\tau}=\frac{c(\tau)-I}{\tau}\varphi+\frac{s(\tau)}{\tau}\psi+\frac{1}{\tau}\int_0^{\tau}s(\tau-\lambda)f(\lambda)d\lambda \tag{2.8}$$

yazılabilir. (2.8) denkleminden açıkça görüldüğü üzere $u'(0)$ ın yaklaşımını elde edebilmek için operatör fonksiyonları

$s(\tau)$, $c(\tau)$ ve $\frac{1}{\tau}\int_0^{\tau}s(\tau-\lambda)f(\lambda)d\lambda$

nın yaklaşım formüllerini elde etmek gerekmektedir. $c(\tau)$, $s(\tau)$ operatör fonksiyonlarının tanımlarından ve e^{-z} fonksiyonu için Pade kesirlerinden yararlanarak (2.1) in üçüncü mertebeden yaklaşımı için

$$\begin{cases} c(\tau) = \frac{R(i\tau B)+R(-i\tau B)}{2} + o(\tau^3), \\ s(\tau) = B^{-1}\frac{R(i\tau B)-R(-i\tau B)}{2i} + o(\tau^3) \end{cases} \tag{2.9}$$

yazılabilir. Burada

$$R(i\tau B) = R = DE, R(-i\tau B) = \tilde{R} = \tilde{D}E \tag{2.10}$$

dur. Benzer şekilde dördüncü mertebeden yaklaşımı için

$$\begin{cases} c(\tau) = \frac{R(i\tau B)+R(-i\tau B)}{2} + o(\tau^4), \\ s(\tau) = B^{-1}\frac{R(i\tau B)-R(-i\tau B)}{2i} + o(\tau^4) \end{cases} \tag{2.11}$$

yazılabilir. Burada

$$R(i\tau B) = R = C\tilde{C}^{-1}, R(-i\tau B) = \tilde{R} = \tilde{C}C^{-1} \tag{2.12}$$

dir (Ashyralyev ve Sobolevskii 2004). Yukarıdaki formüllerde

$B = A^{1/2}$,

$$D=\left(I-\frac{1}{3}\tau^2A+i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\right),\ \widehat{D}=\left(I-\frac{1}{3}\tau^2A-i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\right),$$

$$E=\left(I+\frac{1}{6}\tau^2A+\frac{1}{12}\tau^4A^2\right)^{-1},\ C=\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right),\widehat{C}=\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)$$

dir. (2.9), (2.10), (2.11), (2.12) formüllerinden yararlanarak, $s(\tau)$ ve $c(\tau)$ operatör fonksiyonlarının üçüncü mertebeden yaklaşım formülleri

$$c_\tau(\tau)\ =\ \left(I-\frac{1}{3}\tau^2A\right)\left(I+\frac{1}{6}\tau^2A+\frac{1}{12}\tau^4A^2\right)^{-1}, \tag{2.13}$$

$$s_\tau(\tau)\ =\ \tau\sqrt{I+\frac{1}{72}\tau^4A^2}\left(I+\frac{1}{6}\tau^2A+\frac{1}{12}\tau^4A^2\right)^{-1} \tag{2.14}$$

ve dördüncü mertebeden yaklaşım formülleri

$$c_\tau(\tau)\ =\ \left(I-\frac{5}{12}\tau^2A+\frac{1}{144}\tau^4A^2\right)\left(I+\frac{1}{12}\tau^2A+\frac{1}{144}\tau^4A^2\right)^{-1}, \tag{2.15}$$

$$s_\tau(\tau)=\tau\left(I-\frac{1}{12}\tau^2A\right)\left(I+\frac{1}{12}\tau^2A+\frac{1}{144}\tau^4A^2\right)^{-1} \tag{2.16}$$

elde edilir. Başlangıç şartı $u^{'}(0)$ ın yaklaşım formülünü elde ederken, $f_{1,1}^3$ ve $f_{1,1}^4$ ün

$$\frac{1}{\tau}\int_0^\tau s(\tau-\lambda)f(\lambda)d\lambda-f_{1,1}^3\ =\ o(\tau^3), \tag{2.17}$$

$$\frac{1}{\tau}\int_0^\tau s(\tau-\lambda)f(\lambda)d\lambda-f_{1,1}^4\ =\ o(\tau^4), \tag{2.18}$$

olacak şekilde oluşturulmasına ve formüllerinin yeterince basit olmasına dikkat edilmelidir. $f_{1,1}^3$ ve $f_{1,1}^4$ ün seçimleri bir tektir. Taylor metodu kullanılır ve integrasyon yapılırsa, sırasıyla üçüncü mertebeden yaklaşım formülü

$$f_{1,1}^{3}=\left\{S_{\tau}(\tau)f(0)+\left(-C_{\tau}(\tau)f(0)+S_{\tau}(\tau)f^{'}(0)\right)\frac{\tau}{2}-2C_{\tau}(\tau)f^{'}(0)\frac{\tau^{2}}{6}\right\}$$

$$=\left(I+\frac{\tau^{2}}{12}A+\frac{\tau^{4}}{144}A^{2}\right)^{-1}\left\{\tau f(0)+\left(-f(0)+\tau f^{'}(0)\right)\frac{\tau}{2}-2f^{'}(0)\frac{\tau^{2}}{6}\right\} \tag{2.19}$$

ve dördüncü mertebeden yaklaşım formülü

$$\begin{cases} f_{1,1}^{4}=\left\{S_{\tau}(\tau)f(0)+\left(-C_{\tau}(\tau)f(0)+S_{\tau}(\tau)f^{'}(0)\right)\frac{\tau}{2}\right. \\ +\left(-AS_{\tau}(\tau)f(0)-2C_{\tau}(\tau)f^{'}(0)+S_{\tau}(\tau)f^{''}(0)\right)\frac{\tau^{2}}{6} \\ \left.+\left(AC_{\tau}(\tau)f(0)-3C_{\tau}(\tau)f^{''}(0)\right)\frac{\tau^{3}}{24}\right\}=\left(I+\frac{\tau^{2}}{12}A+\frac{\tau^{4}}{144}A^{2}\right)^{-1}\left\{\tau\left(I-\frac{\tau^{2}A}{12}\right)f(0)\right. \\ \left.+\left(-\left(I-\frac{5\tau^{2}A}{12}\right)f(0)+\tau f^{'}(0)\right)\frac{\tau}{2}+\left(-A\tau f(0)-2f^{'}(0)+\tau f^{''}(0)\right)\frac{\tau^{2}}{6}+\left(Af(0)-3f^{''}(0)\right)\frac{\tau^{3}}{24}\right\} \end{cases} \tag{2.20}$$

elde edilir. İlerki bölümlerde,

$$f_{1,1}=\left\{f(0)+(-f(0)+\tau f^{'}(0))\frac{1}{2}-2f^{'}(0)\frac{\tau}{6}\right\}, \tag{2.21}$$

ve

$$f_{2,2}=\left\{\left(I-\frac{\tau^{2}A}{12}\right)f(0)+\left(-\left(I-\frac{5\tau^{2}A}{12}\right)f(0)+\tau f^{'}(0)\right)\frac{1}{2}\right.$$

$$\left.+\left(-A\tau f(0)-2f^{'}(0)+\tau f^{''}(0)\right)\frac{\tau}{6}+\left(Af(0)-3f^{''}(0)\right)\frac{\tau^{2}}{24}\right\} \tag{2.22}$$

ifadeleri için, $f_{1,1}^{3}=\tau\left(I+\frac{\tau^{2}}{12}A+\frac{\tau^{4}}{144}A^{2}\right)^{-1}f_{1,1}$ ve $f_{1,1}^{4}=\tau\left(I+\frac{\tau^{2}}{12}A+\frac{\tau^{4}}{144}A^{2}\right)^{-1}f_{2,2}$

eşitliklerinden faydalanılacaktır. Sonuç olarak $u'(0)$ türevinin yaklaşımı için

$\dfrac{u_1-u_0}{\tau}=\dfrac{c_\tau(\tau)-I}{\tau}\varphi+\dfrac{s_\tau(\tau)}{\tau}\psi+f_{1,1}^m,\ (m=3 \text{ veya } 4)$ genel formülü yazılabilir.

2.2 Hiperbolik Tip Cauchy Probleminin Sayısal Çözümü için Üçüncü Mertebeden Doğruluk Fark Şeması

Bu kısımda, başlangıç değer problemi (2.1) in yaklaşık çözümü için, formüller (2.6) ve (2.19) dan yararlanarak üçüncü mertebeden doğruluk fark şeması

$$\begin{cases}\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{2}{3}Au_k+\frac{1}{6}A\left(u_{k+1}+u_{k-1}\right)\\ +\frac{1}{12}\tau^2A^2u_{k+1}=f_k,f_k=\frac{2}{3}f(t_k)+\frac{1}{6}\left(f\left(t_{k+1}\right)+f\left(t_{k-1}\right)\right)\\ -\frac{1}{12}\tau^2\left(-Af\left(t_{k+1}\right)+f''(t_{k+1})\right),1\le k\le N-1,\\ u_0=\varphi,\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)\tau^{-1}\left(u_1-u_0\right)\\ =-\left(\frac{\tau}{2}A+\frac{\tau^3}{12}A^2\right)\varphi+\sqrt{I+\frac{1}{72}\tau^4A^2}\psi+\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)f_{1,1}^3\end{cases} \tag{2.23}$$

elde edildi. Bununla beraber, fark şemalarının uygulamalarında (realization) önce $A^{1/2}$ operatörünün inşa edilmesi gereklidir. Uygulamada $A^{1/2}$ nin elde edilmesi zordur. Bu sebeple, fark şemasının, başlangıç değer problemlerindeki teorik önemine rağmen, sayısal uygulamalardaki işlevselliği hiç de iyi değildir. Dolayısıyla, bu tezde problem (2.1) in yaklaşık çözümü için, A operatörünün tam kuvvetlerinden yararlanılarak, üçüncü mertebeden doğruluk fark şeması ve bu fark şemalarının çözümleri için kararlılık kestirimleri elde edildi. Denklem (2.6) ve (2.19) dan yararlanarak üçüncü mertebeden iki adımlı yaklaşımlı fark şeması

$$\begin{cases}\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{2}{3}Au_k+\frac{1}{6}A(u_{k+1}+u_{k-1})\\ +\frac{1}{12}\tau^2A^2u_{k+1}=f_k,f_k=\frac{2}{3}f(t_k)+\frac{1}{6}(f(t_{k+1})+f(t_{k-1}))\\ -\frac{1}{12}\tau^2(-Af(t_{k+1})+f''(t_{k+1})),\ 1\le k\le N-1,\\ u_0=\varphi,\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\tau^{-1}(u_1-u_0)\\ =-\frac{\tau}{2}A\varphi+\left(I-\frac{\tau^2}{12}A\right)\psi+f_{1,1}\tau,\end{cases} \tag{2.24}$$

elde edildi. Burada

$$f_{1,1} = \left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)\tau^{-1}f_{1,1}^3 = \left\{f(0) + (-f(0) + \tau f'(0))\frac{1}{2} - 2f'(0)\frac{\tau}{6}\right\}$$

dır.

Öncelikle ispat aşamalarında yararlanılacak bir lemma ele alalım. Bu kısımda ayrıca, (2.10) da verilen R, $\tilde{R}$ operatörlerinden ve aşağıdaki operatörlerden yararlanıldı. Bu operatörler

$$R_1 = \left(-\frac{5\tau^4}{144}A^2 + \frac{7\tau^6}{216}A^3 - i\tau A^{1/2}\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)\sqrt{I + \frac{1}{72}\tau^4A^2}\right)$$

$$\times\left(-i\tau A^{1/2}\left(\sqrt{I + \frac{1}{72}\tau^4A^2}\right)\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)\right)^{-1}, \tag{2.25}$$

ve eşleniği $\tilde{R}_1$,

$$R_2 = \left(I - \frac{\tau^2}{12}A\right)\left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2\right)$$

$$\times\left(-iA^{1/2}\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)\sqrt{I + \frac{1}{72}\tau^4A^2}\right)^{-1}, \tag{2.26}$$

$$R_3 = \left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2\right)\left(\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)\left(-i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4A^2}\right)\right)^{-1}, \tag{2.27}$$

$$R_4 = \left(I + \frac{\tau^2}{3}A + \frac{\tau^4}{9}A^2 + \frac{\tau^6}{72}A^3\right)$$

$$\times\left(-iA^{1/2}\left(\sqrt{I + \frac{1}{72}\tau^4A^2}\right)\left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2\right)\left(I + \frac{\tau^2}{6}A\right)\right)^{-1}, \tag{2.28}$$

$$R_5 = \left(-\frac{\tau^2}{2}A - \frac{\tau^4}{12}A^2 + i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\right)\left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2\right)^{-1}, \quad (2.29)$$

ve eşleniği $\widetilde{R}_5$,

$$R_6 = \left(I - \frac{1}{3}\tau^2 A + i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\right)$$

$$\times\left(\frac{\tau^2}{2}A + \frac{\tau^4}{12}A^2 - i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\right)^{-1}, \quad (2.30)$$

ve eşleniği $\widehat{R}_6$ dır.

Lemma 2.1 Aşağıdaki kararlılık kestirimleri sağlanır.

$$\begin{cases} \|R\|_{H\mapsto H} \leq 1, \|\tilde{R}\|_{H\mapsto H} \leq 1, \\ \|R_1\|_{H\mapsto H} \leq 1, \|\tilde{R}_1\|_{H\mapsto H} \leq 1, \\ \|A^{1/2}R_2\|_{H\mapsto H} \leq 1, \|\tau A^{1/2}R_3\|_{H\mapsto H} \leq 1, \\ \|A^{1/2}R_4\|_{H\mapsto H} \leq 1, \|A^{-1/2}R_5\|_{H\mapsto H} \leq \tau, \\ \left\| A^{-1/2}\widetilde{R}_5 \right\|_{H\mapsto H} \leq \tau, \|\tau A^{1/2}R_6\|_{H\mapsto H} \leq 1, \\ \left\| \tau A^{1/2}\widetilde{R}_6 \right\|_{H\mapsto H} \leq 1. \end{cases} \quad (2.31)$$

İspat: Kendisine eşlenik, pozitif tanımlı operatörün spektral özeliği yardımıyla

$$\left\| \left(I - \frac{1}{3}\tau^2 A \pm i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\right)\left(I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2\right)^{-1} \right\|_{H\mapsto H}$$

$$\leq \sup_{\delta\leq\lambda<\infty}\left|\frac{1-\frac{1}{3}\tau^2\lambda + i\tau\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}}{1+\frac{1}{6}\tau^2\lambda+\frac{1}{12}\tau^4\lambda^2}\right|$$

yazılabilir. Buradan,

$$\frac{\left(1-\frac{1}{3}\tau^2\lambda\right)^2+\left(\tau\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)^2}{\left(1+\frac{1}{6}\tau^2\lambda+\frac{1}{12}\tau^4\lambda^2\right)^2}$$

$$=\frac{1+\frac{1}{3}\tau^2\lambda+\frac{1}{9}\tau^4\lambda^2+\frac{1}{72}\tau^6\lambda^3}{1+\frac{1}{3}\tau^2\lambda+\frac{7}{36}\tau^4\lambda^2+\frac{1}{36}\tau^6\lambda^3+\frac{1}{144}\tau^8\lambda^4}\leq 1$$

olduğundan, $\|R\|_{H\mapsto H}\leq 1$ ve $\|\tilde{R}\|_{H\mapsto H}\leq 1$ dir. Benzer şekilde,

$$\Bigg\|\left(-\frac{5\tau^4}{144}A^2+\frac{7\tau^6}{216}A^3\pm i\tau A^{1/2}\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\sqrt{I+\frac{1}{72}\tau^4A^2}\right)$$

$$\times\left(-i\tau A^{1/2}\left(\sqrt{I+\frac{1}{72}\tau^4A^2}\right)\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\right)^{-1}\Bigg\|_{H\mapsto H}$$

$$\leq \sup_{\delta\leq\lambda<\infty}\left|\frac{\left(-\frac{5\tau^4}{144}\lambda^2+\frac{7\tau^6}{216}\lambda^3\pm i\tau\lambda^{1/2}\left(1+\frac{\tau^2}{12}\lambda+\frac{\tau^4}{144}\lambda^2\right)\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)}{i\tau\lambda^{1/2}\left(\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)\left(1+\frac{\tau^2}{12}\lambda+\frac{\tau^4}{144}\lambda^2\right)}\right|$$

yazılabilir. Buradan,

$$\frac{\left(-\frac{5\tau^4}{144}\lambda^2+\frac{7\tau^6}{216}\lambda^3\right)^2+\left(\tau\lambda^{1/2}\left(1+\frac{\tau^2}{12}\lambda+\frac{\tau^4}{144}\lambda^2\right)\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)^2}{\left(\tau\lambda^{1/2}\left(\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)\left(1+\frac{\tau^2}{12}\lambda+\frac{\tau^4}{144}\lambda^2\right)\right)^2}\leq 1$$

olduğundan, $\|R_1\|_{H\mapsto H}\leq 1$ ve $\|\widehat{R}_1\|_{H\mapsto H}\leq 1$ dir. Aynı şekilde

$$\Bigg\|A^{1/2}\left(I-\frac{\tau^2}{12}A\right)\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)$$

$$\times\left(-iA^{1/2}\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\sqrt{I+\frac{1}{72}\tau^4A^2}\right)^{-1}\Bigg\|_{H\to H}$$

$$\leq \sup_{\delta\leq\lambda<\infty}\left|\frac{\lambda^{1/2}\left(1+\frac{\tau^2}{12}\lambda-\frac{5\tau^4}{72}\lambda^2-\frac{\tau^6}{144}\lambda^3\right)}{i\lambda^{1/2}\left(1+\frac{\tau^2}{12}\lambda+\frac{\tau^4}{144}\lambda^2\right)\sqrt{1+\frac{1}{72}\tau^4\lambda^2}}\right|$$

yazılabilir. Buradan da

$$\frac{\lambda\left(1+\frac{\tau^2}{12}\lambda-\frac{5\tau^4}{72}\lambda^2-\frac{\tau^6}{144}\lambda^3\right)^2}{\left(\lambda^{1/2}\left(1+\frac{\tau^2}{12}\lambda+\frac{\tau^4}{144}\lambda^2\right)\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)^2}\leq 1$$

olduğundan, $\left\|A^{1/2}R_2\right\|_{H\mapsto H}\leq 1$ dir. Aynı yolla

$$\left\|\tau A^{1/2}\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)\left(i\tau A^{1/2}\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\sqrt{I+\frac{1}{72}\tau^4A^2}\right)^{-1}\right\|_{H\to H}$$

$$\leq \sup_{\delta\leq\lambda<\infty}\left|\frac{\tau\lambda^{1/2}\left(1+\frac{\tau^2}{6}\lambda+\frac{\tau^4}{12}\lambda^2\right)}{i\tau\lambda^{1/2}\left(1+\frac{\tau^2}{12}\lambda+\frac{\tau^4}{144}\lambda^2\right)\sqrt{1+\frac{1}{72}\tau^4\lambda^2}}\right|$$

yazılabilir. Burada

$$\frac{\tau^2\lambda\left(1+\frac{\tau^2}{6}\lambda+\frac{\tau^4}{12}\lambda^2\right)^2}{\left(\tau\lambda^{1/2}\left(1+\frac{\tau^2}{12}\lambda+\frac{\tau^4}{144}\lambda^2\right)\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)^2}\leq 1$$

olduğundan, $\left\|\tau A^{1/2}R_3\right\|_{H\mapsto H}\leq 1$ dir. Benzer adımlar kullanılarak

$$\left\|\left(I+\frac{\tau^2}{3}A+\frac{\tau^4}{9}A^2+\frac{\tau^6}{72}A^3\right)\right.$$

$$\left.\times\left(-iA^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)\left(I+\frac{\tau^2}{6}A\right)\right)^{-1}\right\|_{H\mapsto H}$$

$$\leq \sup_{\delta\leq\lambda<\infty}\left|\frac{1+\frac{\tau^2}{3}\lambda+\frac{\tau^4}{9}\lambda^2+\frac{\tau^6}{72}\lambda^3}{i\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\left(1+\frac{\tau^2}{6}\lambda+\frac{\tau^4}{12}\lambda^2\right)\left(1+\frac{\tau^2}{6}\lambda\right)}\right|$$

yazılabilir. Buradan,

$$\frac{\left(1+\frac{\tau^2}{3}\lambda+\frac{\tau^4}{9}\lambda^2+\frac{\tau^6}{72}\lambda^3\right)^2}{\left(\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\left(1+\frac{\tau^2}{6}\lambda+\frac{\tau^4}{12}\lambda^2\right)\left(1+\frac{\tau^2}{6}\lambda\right)\right)^2}\leq 1$$

olduğundan, $\left\|A^{1/2}R_4\right\|_{H\mapsto H}\leq 1$ dir. Aynı şekilde

$$\left\|\left(-\frac{\tau^2}{2}A-\frac{\tau^4}{12}A^2\pm i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\right)\left(A^{1/2}\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)\right)^{-1}\right\|_{H\mapsto H}$$

$$\leq \sup_{\delta\leq\lambda<\infty}\left|\frac{-\frac{\tau^2}{2}\lambda-\frac{\tau^4}{12}\lambda^2\pm i\tau\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}}{\lambda^{1/2}\left(1+\frac{1}{6}\tau^2\lambda+\frac{1}{12}\tau^4\lambda^2\right)}\right|$$

eşitsizliği yazılabilir. Buradan da

$$\frac{\left(-\frac{\tau^2}{2}\lambda-\frac{\tau^4}{12}\lambda^2\right)^2+\left(\tau\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)^2}{\lambda\left(1+\frac{1}{6}\tau^2\lambda+\frac{1}{12}\tau^4\lambda^2\right)^2}$$

$$=\frac{\tau^2\lambda+\frac{\tau^4}{4}\lambda^2+\frac{7\tau^6}{72}\lambda^3+\frac{\tau^8}{144}\lambda^4}{\lambda+\frac{1}{3}\tau^2\lambda^2+\frac{7}{36}\tau^4\lambda^3+\frac{\tau^6}{36}\lambda^4+\frac{\tau^8}{144}\lambda^5}\leq\tau^2$$

olduğundan, $\left\|A^{-1/2}R_5\right\|_{H\mapsto H}\leq\tau$ ve $\left\|A^{-1/2}\widehat{R}_5\right\|_{H\mapsto H}\leq\tau$ dur. Tamamen aynı metotla

$$\left\|\tau A^{1/2}\left(I-\frac{1}{3}\tau^2A+i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\right)\right.$$

$$\times\left(\frac{\tau^2}{2}A+\frac{\tau^4}{12}A^2-i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\right)^{-1}\Bigg\|_{H\mapsto H}$$

$$\leq \sup_{\delta\leq\lambda<\infty}\left|\frac{\tau\lambda^{1/2}\left(1-\frac{1}{3}\tau^2\lambda+i\tau\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)}{\frac{\tau^2}{2}\lambda+\frac{\tau^4}{12}\lambda^2-i\tau\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}}\right|$$

yazılabilir. Burada

$$\frac{\tau^2\lambda\left(1-\frac{1}{3}\tau^2\lambda\right)^2+\tau^2\lambda\left(\tau\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)^2}{\left(\frac{\tau^2}{2}\lambda+\frac{\tau^4}{12}\lambda^2\right)^2+\left(\tau\lambda^{1/2}\sqrt{1+\frac{1}{72}\tau^4\lambda^2}\right)^2}$$

$$=\frac{\lambda\left(1+\frac{1}{3}\tau^2\lambda+\frac{1}{9}\tau^4\lambda^2+\frac{1}{72}\tau^6\lambda^3\right)}{\lambda+\frac{1}{4}\tau^2\lambda^2+\frac{7}{72}\tau^4\lambda^3+\frac{1}{144}\tau^6\lambda^4}$$

$$=\frac{1+\frac{1}{3}\tau^2\lambda+\frac{1}{9}\tau^4\lambda^2+\frac{1}{72}\tau^6\lambda^3}{1+\frac{1}{4}\tau^2\lambda+\frac{7}{72}\tau^4\lambda^2+\frac{1}{144}\tau^6\lambda^3}\leq 1$$

olduğundan, $\left\|\tau A^{1/2}R_6\right\|_{H\mapsto H}\leq 1$ ve $\left\|\tau A^{1/2}\widehat{R}_6\right\|_{H\mapsto H}\leq 1$ dir. Bu Lemma 2.1 i ispat eder.

Teorem 2.1 $\varphi\in D(A)$, $\psi\in D(A^{\frac{1}{2}})$, $f_{1,1}\in D(A^{\frac{1}{2}})$ olsun. Bu takdirde, fark şeması (2.24) ün çözümü için aşağıdaki kararlılık kestirimleri sağlanır;

$$\max_{1\leq k\leq N}\|u_k\|_H\leq M\left\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\|\varphi\|_H+\|A^{-1/2}\psi\|_H+\tau\|A^{-1/2}f_{1,1}\|_H\right\}, \tag{2.32}$$

$$\max_{1\leq k\leq N}\|A^{1/2}u_k\|_H+\max_{1\leq k\leq N}\left\|\frac{u_k-u_{k-1}}{\tau}\right\|_H$$

$$\leq M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|A^{1/2}\varphi\|_H+\|\psi\|_H+\tau\|f_{1,1}\|_H\right\}, \tag{2.33}$$

$$\max_{1\le k\le N} \|Au_k\|_H + \max_{1\le k\le N} \left\| A^{1/2} \frac{u_k - u_{k-1}}{\tau} \right\|_H + \max_{1\le k\le N-1} \left\| \frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} \right\|_H$$

$$\le M \left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A\varphi\|_H + \|A^{1/2}\psi\|_H + \tau \|A^{1/2} f_{1,1}\|_H \right\}. \qquad (2.34)$$

Burada M sabiti τ, φ, ψ, $f_{1,1}$ ve f_s $(1 \le s \le N-1)$ den bağımsızdır.

İspat: İlk olarak problem (2.1) in çözümü için formül elde edelim. Başlangıç değer problemi

$$au_{k-1} - cu_k + bu_{k+1} = \varphi_k, 1 \le k \le N-1, u_0 = \varphi, u_1 = \psi \qquad (2.35)$$

in çözümünün tekliği açıktır. Problem (2.35) in çözümü için aşağıdaki formül

$$u_0 = \varphi, u_1 = \psi,$$

$$u_k = R\tilde{R}(\tilde{R} - R)^{-1}\left[R^{k-1} - \tilde{R}^{k-1}\right]\varphi + (\tilde{R} - R)^{-1}(\tilde{R}^k - R^k)\psi$$
$$+\sum_{j=1}^{k-1} R\tilde{R}\left((\tilde{R} - R)a\right)^{-1}\left[\tilde{R}^{k-j} - R^{k-j}\right]\varphi_j, \qquad (2.36)$$

yazılabilir (Ashyralyev ve Sobolevskii 2004). Burada q_1, q_2 denklem (2.35) in kökleridir ve

$$R = q_1 = \tfrac{c+\sqrt{c^2-4ab}}{2b},\ \tilde{R} = q_2 = \tfrac{c-\sqrt{c^2-4ab}}{2b},\ (a \ne 0, b \ne 0)$$

dir. Problem (2.35) yeniden düzenlenirse

$$\left(I + \frac{\tau^2}{6}A\right)u_{k-1} - \left(2 - \frac{2\tau^2}{3}A\right)u_k + \left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2\right)u_{k+1} = \tau^2 f_k, 1 \le k \le N-1,$$

$$u_0=\varphi,\ u_1=\varphi+\tau\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)^{-1}$$

$$\times\left(-\frac{\tau}{2}A\varphi+\left(I-\frac{\tau^2}{12}A\right)\psi+\tau f_{1,1}\right)$$

yazılabilir. Denklemde, a ile $\left(I+\frac{\tau^2}{6}A\right)$, b ile $\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)$, c ile $\left(2-\frac{2\tau^2}{3}A\right)$, ψ ile

$$\varphi+\tau\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)^{-1}\left(-\frac{\tau}{2}A\varphi+\left(I-\frac{\tau^2}{12}A\right)\psi+\tau f_{1,1}\right),$$

φ_k ile $\tau^2 f_k$ değiştirilerek ve formül (2.36) uygulanarak, u_k $(2\le k\le N)$ için

$$u_k=\frac{1}{2}\left[\widehat{R}_1R^k-R_1\tilde{R}^k\right]\varphi+\frac{1}{2}\left[\tilde{R}^k-R^k\right]R_2\psi+\frac{1}{2}\left[\tilde{R}^k-R^k\right]R_3\tau^2 f_{1,1}$$

$$+\frac{1}{2}R_4\sum_{s=1}^{k-1}\left[\tilde{R}^{k-s}-R^{k-s}\right]f_s\tau^2 \tag{2.37}$$

formülü elde edilir. Şimdi (2.32), (2.33), (2.34) kestirimlerini elde edelim. u_1 başlangıç şartı için yukarıdaki formülden, (2.31) deki kestirimler ve aşağıdaki

$$\left\|\tau A^{1/2}\left(I-\frac{\tau^2}{12}A\right)\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)^{-1}\right\|_{H\to H}\le 12, \tag{2.38}$$

$$\left\|\tau A^{1/2}\left(I+\frac{1}{12}\tau^2A+\frac{1}{144}\tau^4A^2\right)^{-1}\right\|_{H\to H}\le\frac{12\sqrt{11}}{12+\sqrt{11}}, \tag{2.39}$$

basit kestirimlerinden yararlanarak,

$$\|u_1\|_H\le\left\|\left(I-\frac{5}{12}\tau^2A+\frac{\tau^4}{144}A^2\right)\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)^{-1}\right\|_{H\to H}\|\varphi\|_H$$

$$+\left\|\tau A^{1/2}\left(I-\frac{\tau^2}{12}A\right)\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)^{-1}\right\|_{H\to H}\|A^{-1/2}\psi\|_H$$

$$+\left\|\tau A^{1/2}\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)^{-1}\right\|_{H\to H}\tau\|A^{-1/2}f_{1,1}\|_H$$

$$\leq \|\varphi\|_H+12\|A^{-1/2}\psi\|_H+\frac{12\sqrt{11}}{12+\sqrt{11}}\tau\|A^{-1/2}f_{1,1}\|_H$$

$$\leq M\left\{\sum_{s=1}^{N-1}\left\|A^{-1/2}f_s\right\|_H\tau+\left\|\varphi\right\|_H+\left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{1,1}\right\|_H\right\}$$

kararlılık kestirimi elde edilir. Şimdi u_1 formülüne $A^{1/2}$ operatörü uygulanarak ve (2.31), (2.38), (2.39) kestirimlerinden yararlanarak

$$\|A^{1/2}u_1\|_H\leq\left\|\left(I-\frac{5\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\|A^{1/2}\varphi\|_H$$

$$+\left\|\tau A^{1/2}\left(I-\frac{\tau^2}{12}A\right)\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)^{-1}\right\|_{H\to H}\|\psi\|_H$$

$$+\left\|\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|f_{1,1}\|_H\leq\|A^{1/2}\varphi\|_H+12\|\psi\|_H$$

$$+\frac{12\sqrt{11}}{12+\sqrt{11}}\tau\left\|f_{1,1}\right\|_H\leq M\left\{\sum_{s=1}^{N-1}\left\|f_s\right\|_H\tau+\left\|A^{1/2}\varphi\right\|_H+\left\|\psi\right\|_H+\tau\left\|f_{1,1}\right\|_H\right\}$$

kararlılık kestirimi elde edilir. Keza

$$\frac{u_1-u_0}{\tau}=\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}$$

$$\times\left(-\frac{\tau}{2}A\varphi+\left(I-\frac{\tau^2}{12}A\right)\psi+\tau f_{1,1}\right) \qquad (2.41)$$

formülü, (2.31), (2.40) ve

$$\left\| \left(I - \frac{\tau^2 A}{12}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \leq 1, \tag{2.42}$$

$$\left\| \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \leq 1, \tag{2.43}$$

basit kestirimleri kullanılarak,

$$\left\| \frac{u_1 - u_0}{\tau} \right\|_H \leq \left\| \frac{1}{2}\tau A^{1/2}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \|A^{1/2}\varphi\|_H$$

$$+ \left\| \left(I - \frac{\tau^2}{12}A\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \|\psi\|_H$$

$$+ \left\| \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \tau \|f_{1,1}\|_H \leq \frac{6\sqrt{11}}{12+\sqrt{11}} \|A^{1/2}\varphi\|_H + \|\psi\|_H + \tau\|f_{1,1}\|_H$$

$$\leq M\left\{ \sum_{s=1}^{N-1} \|f_s\|_H \tau + \left\|A^{1/2}\varphi\right\|_H + \|\psi\|_H + \tau\left\|f_{1,1}\right\|_H \right\} \tag{2.44}$$

elde edilir. Formül (2.41) e $A^{1/2}$ operatörü uygulanarak ve (2.31), (2.40), (2.42), (2.43) kestirimlerinden yararlanarak

$$\left\|A^{1/2} \frac{u_1 - u_0}{\tau} \right\|_H \leq \left\| \frac{\tau}{2} A^{1/2}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \|A\varphi\|_H$$

$$+ \left\| \left(I - \frac{\tau^2}{12}A\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \|A^{1/2}\psi\|_H$$

$$+\left\| \left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \tau \| A^{1/2} f_{1,1} \|_H$$

$$\le \frac{6\sqrt{11}}{12+\sqrt{11}} \| A\varphi \|_H + \| A^{1/2}\psi \|_H + \tau \| A^{1/2} f_{1,1} \|_H$$

$$\le M \left\{ \sum_{s=2}^{N-1} \left\| f_s - f_{s-1} \right\|_H + \left\| f_1 \right\|_H + \left\| A\varphi \right\|_H + \left\| A^{1/2}\psi \right\|_H + \tau \left\| A^{1/2} f_{1,1} \right\|_H \right\} \quad (2.45)$$

kararlılık kestirimi bulunur. Benzer şekilde, u_1 e A operatörü uygulanarak ve (2.31), (2.38), (2.39), (2.40) kestirimleri kullanılarak

$$\| Au_1 \|_H \le \left\| \left(I-\frac{5\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \| A\varphi \|_H$$

$$+\left\| \tau A^{1/2}\left(I-\frac{\tau^2}{12}A\right)\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)^{-1} \right\|_{H\to H} \| A^{1/2}\psi \|_H$$

$$+\left\| \tau A^{1/2}\left(I+\frac{1}{12}\tau^2 A+\frac{1}{144}\tau^4 A^2\right)^{-1} \right\|_{H\to H} \tau \| A^{1/2} f_{1,1} \|_H$$

$$\le \| A\varphi \|_H + 12 \| A^{1/2}\psi \|_H + \frac{12\sqrt{11}}{12+\sqrt{11}} \tau \| A^{1/2} f_{1,1} \|_H$$

$$\le M \left\{ \sum_{s=2}^{N-1} \left\| f_s - f_{s-1} \right\|_H + \left\| f_1 \right\|_H + \left\| A\varphi \right\|_H + \left\| A^{1/2}\psi \right\|_H + \tau \left\| A^{1/2} f_{1,1} \right\|_H \right\} \quad (2.46)$$

kararlılık kestirimi elde edilir. Benzer adımla, fark şeması (2.44) ve formül (2..41) kullanılarak

$$\frac{u_2 - 2u_1 + u_0}{\tau^2} = \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1}$$

$$\times \left\{ -\left(I - \frac{\tau^2 A}{3} - \frac{5\tau^4 A^2}{72} + \frac{\tau^6 A^3}{1728}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} A\varphi \right.$$

$$- \tau\left(A + \frac{1}{6}\tau^2 A^2\right)\left(I - \frac{\tau^2}{12}A\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1}\psi$$

$$\left. -\left(A + \frac{\tau^2 A^2}{6}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1}\tau^2 f_{1,1} + f_1 \right\} \qquad (2.47)$$

elde edilir. Formül (2.47), kararlılık kestirimleri (2.31), (2.39), (2.40), (2.43) ve

$$\left\| \left(I - \frac{\tau^2 A}{3} - \frac{5\tau^4 A^2}{72} + \frac{\tau^6 A^3}{1728}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-2} \right\|_{H\to H} \leq 2, \qquad (2.48)$$

$$\left\| \left(I + \frac{\tau^2 A}{6}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \leq 2, \qquad (2.49)$$

basit kestirimleri yardımı ile

$$\left\| \frac{u_2 - 2u_1 + u_0}{\tau^2} \right\|_H$$

$$\leq \left\| \left(I - \frac{\tau^2 A}{3} - \frac{5\tau^4 A^2}{72} + \frac{\tau^6 A^3}{1728}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-2} \right\|_{H\to H} \| A\varphi \|_H$$

$$+ \left\| \left(I + \frac{\tau^2 A}{6}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \left\| \tau A^{1/2}\left(I - \frac{\tau^2}{12}A\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \left\| A^{1/2}\psi \right\|_H$$

$$+\left\|\left(I+\frac{\tau^2A}{6}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|A^{1/2}f_{1,1}\right\|_H\tau$$

$$+\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|f_1\right\|_H\le 2\left\|A\varphi\right\|_H+\left\|f_1\right\|_H+24\left\|A^{1/2}\psi\right\|_H+\frac{12\sqrt{11}}{12+\sqrt{11}}\tau\left\|A^{1/2}f_{1,1}\right\|_H$$

$$\le M\left\{\sum_{s=2}^{N-1}\left\|f_s-f_{s-1}\right\|_H+\left\|f_1\right\|_H+\left\|A\varphi\right\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\} \qquad (2.50)$$

yazılır. Sonuç olarak, $k=1$ hali için aşağıdaki kestirimler

$$\|u_k\|_H\le M\left\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\|\varphi\|_H+\|A^{-1/2}\psi\|_H+\tau\|A^{-1/2}f_{1,1}\|_H\right\},$$

$$\|A^{1/2}u_k\|_H\le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|A^{1/2}\varphi\|_H+\|\psi\|_H+\tau\|f_{1,1}\|_H\right\},$$

$$\left\|\frac{u_k-u_{k-1}}{\tau}\right\|_H\le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|A^{1/2}\varphi\|_H+\|\psi\|_H+\tau\|f_{1,1}\|_H\right\},$$

$$\left\|A^{1/2}\frac{u_k-u_{k-1}}{\tau}\right\|_H\le M\left\{\sum_{s=2}^{N-1}\left\|f_s-f_{s-1}\right\|_H+\left\|f_1\right\|_H+\left\|A\varphi\right\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\},$$

$$\left\|Au_k\right\|_H\le M\left\{\sum_{s=2}^{N-1}\left\|f_s-f_{s-1}\right\|_H+\left\|f_1\right\|_H+\left\|A\varphi\right\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\},$$

$$\left\|\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\|_H\le M\left\{\sum_{s=2}^{N-1}\left\|f_s-f_{s-1}\right\|_H+\left\|f_1\right\|_H+\left\|A\varphi\right\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}$$

elde edilir.

Şimdi yukarıdaki kararlılık kestirimlerini tüm $k \geq 2$ değerleri için ispat edelim. Formül (2.37) ve (2.31) de verilen kestirimler kullanılır, üçgen eşitsizliği uygulanırsa $k \geq 2$ hali için

$$\|u_k\|_H \leq \frac{1}{2}\left(\|\tilde{R}_1\|_{H\to H}\|R^k\|_{H\to H} + \|R_1\|_{H\to H}\|\tilde{R}^k\|_{H\to H}\right)\|\varphi\|_H$$

$$+\frac{1}{2}\left(\|A^{1/2}R_2\|_{H\to H}\|\tilde{R}^k\|_{H\to H} + \|A^{1/2}R_2\|_{H\to H}\|R^k\|_{H\to H}\right)\|A^{-1/2}\psi\|_H$$

$$+\frac{1}{2}\left(\|\tau A^{1/2}R_3\|_{H\to H}\|\tilde{R}^k\|_{H\to H} + \|\tau A^{1/2}R_3\|_{H\to H}\|R^k\|_{H\to H}\right)\tau\|A^{-1/2}f_{1,1}\|_H$$

$$+\frac{1}{2}\|\tau A^{1/2}R_4\|_{H\to H}\sum_{s=1}^{k-1}\left[\|\tilde{R}^{k-s}\|_{H\to H} + \|R^{k-s}\|_{H\to H}\right]\|A^{-1/2}f_s\|_H\tau$$

$$\leq M\left\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau + \|\varphi\|_H + \|A^{-1/2}\psi\|_H + \tau\|A^{-1/2}f_{1,1}\|_H\right\} \qquad (2.51)$$

kararlılık kestirimi elde edilir. Bütün k değerlerinde, $\|u_k\|_H$ için elde edilen kestirimleri birlikte değerlendirilirse, kararlılık kestirimi (2.32) elde edilir. Formül (2.37) ye $A^{1/2}$ operatörü ve üçgen eşitsizliği uygulanır, (2.31) de verilen kestirimlerden yararlanılırsa, $k \geq 2$ hali için

$$\|A^{1/2}u_k\|_H \leq \frac{1}{2}\left(\|\tilde{R}_1\|_{H\to H}\|R^k\|_{H\to H} + \|R_1\|_{H\to H}\|\tilde{R}^k\|_{H\to H}\right)\|A^{1/2}\varphi\|_H$$

$$+\frac{1}{2}\left(\|A^{1/2}R_2\|_{H\to H}\|\tilde{R}^k\|_{H\to H} + \|A^{1/2}R_2\|_{H\to H}\|R^k\|_{H\to H}\right)\|\psi\|_H$$

$$+\frac{1}{2}\left(\|\tau A^{1/2}R_3\|_{H\to H}\|\tilde{R}^k\|_{H\to H} + \|\tau A^{1/2}R_3\|_{H\to H}\|R^k\|_{H\to H}\right)\tau\|f_{1,1}\|_H$$

$$+\frac{1}{2}\|\tau A^{1/2}R_4\|_{H\to H}\sum_{s=1}^{k-1}\left(\|\tilde{R}^{k-s}\|_{H\to H}+\|R^{k-s}\|_{H\to H}\right)\|f_s\|_H\tau$$

$$\leq M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|A^{1/2}\varphi\|_H+\|\psi\|_H+\tau\|f_{1,1}\|_H\right\}$$

kararlılık kestirimi bulunur. Bütün k değerlerinde $\|A^{1/2}u_k\|_H$ için kararlılık kestirimleri birlikte değerlendirilirse

$$\max_{1\leq k\leq N}\left\|A^{1/2}u_k\right\|_H\leq M\left\{\sum_{s=1}^{N-1}\left\|f_s\right\|_H\tau+\|A^{1/2}\varphi\|_H+\|\psi\|_H+\tau\left\|f_{1,1}\right\|_H\right\} \tag{2.52}$$

kararlılık kestirimi elde edilir.

(2.33) formülünü ispat etmek, (2.37) yardımı ile

$$\frac{u_k-u_{k-1}}{\tau}=\frac{1}{\tau}\left\{\frac{1}{2}\left[\widetilde{R}_1R_5R^{k-1}-R_1\widetilde{R}_5\tilde{R}^{k-1}\right]\varphi\right.$$

$$+\frac{1}{2}\left[\widetilde{R}_5\tilde{R}^{k-1}-R_5R^{k-1}\right]R_2\psi+\frac{1}{2}\left[\widetilde{R}_5\tilde{R}^{k-1}-R_5R^{k-1}\right]R_3\tau^2f_{1,1}$$

$$\left.+\frac{1}{2}\left[\tilde{R}-R\right]R_4\tau^2f_{k-1}+\frac{1}{2}R_4\sum_{s=1}^{k-2}\left[\widehat{R}_5\tilde{R}^{k-1-s}-R_5R^{k-1-s}\right]f_s\tau^2\right\} \tag{2.53}$$

yazabilir. Kestirim (2.31) den yararlanarak ve üçgen eşitsizliği uygulanarak, formül (2.53) den

$$\left\|\frac{u_k-u_{k-1}}{\tau}\right\|_H\leq\frac{1}{\tau}\left\{\frac{1}{2}\left(\left\|A^{-1/2}R_5\right\|_{H\to H}\left\|\widehat{R}_1\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right.\right.$$

$$+\left\|A^{-1/2}\widehat{R}5\right\|_{H\to H}\left\|R_1\right\|_{H\to H}\left\|\widehat{R}^{k-1}\right\|_{H\to H}\Big)\left\|A^{1/2}\varphi\right\|_H+\frac{1}{2}\Big(\left\|A^{1/2}R_2\right\|_{H\to H}\left\|A^{-1/2}\widehat{R}_5\right\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}$$

$$+\left\|A^{1/2}R_2\right\|_{H\to H}\left\|A^{-1/2}R_5\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\Big)\|\psi\|_H+\frac{1}{2}\Big(\left\|\tau A^{1/2}R_3\right\|_{H\to H}\left\|A^{-1/2}\widehat{R}5\right\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}$$

$$+\|\tau A^{1/2}R_3\|_{H\to H}\|A^{-1/2}R_5\|_{H\to H}\|R^{k-1}\|_{H\to H}\Big)$$

$$\times\tau\|f_{1,1}\|_H+\frac{1}{2}\Big(\|\tau A^{1/2}R_4\|_{H\to H}\|\tilde{R}\|_{H\to H}+\|\tau A^{1/2}R_4\|_{H\to H}\|R\|_{H\to H}\Big)$$

$$\times\|A^{-1/2}f_{k-1}\|_H\tau+\frac{1}{2}\|\tau A^{1/2}R_4\|_{H\to H}\sum_{s=1}^{k-2}\Big(\|A^{-1/2}\widetilde{R}5\|_{H\to H}\|\tilde{R}^{k-1-s}\|_{H\to H}$$

$$+\|A^{-1/2}R_5\|_{H\to H}\|R^{k-1-s}\|_{H\to H}\Big)\|f_s\|_H\tau\Big\}$$

$$\leq M\left\{\sum_{s=1}^{N-1}\left\|f_s\right\|_H\tau+\left\|A^{1/2}\varphi\right\|_H+\left\|\psi\right\|_H+\tau\left\|f_{1,1}\right\|_H\right\} \tag{2.54}$$

kararlılık kestirimi elde edilir. Bütün k değerlerinde $\|\tau^{-1}(u_k-u_{k-1})\|_H$ için kestirimler birlikte değerlendirilecek olursa

$$\max_{1\leq k\leq N-1}\left\|\frac{u_k-u_{k-1}}{\tau}\right\|_H\leq M\left\{\sum_{s=1}^{N-1}\left\|f_s\right\|_H\tau+\left\|A^{1/2}\varphi\right\|_H+\left\|\psi\right\|_H+\tau\left\|f_{1,1}\right\|_H\right\} \tag{2.55}$$

kararlılık kestirimi bulunur. Böylece kararlılık kestirimleri (2.52) ve (2.55) yardımı ile (2.33) kestirimi elde edilmiş olur.

(2.53) denklemine Abel formülü uygulanarak

$$\frac{u_k-u_{k-1}}{\tau}=\frac{1}{\tau}\left\{\frac{1}{2}\left[\widetilde{R}_1R_5R^{k-1}-R_1\widetilde{R}_5\tilde{R}^{k-1}\right]\varphi\right.$$

$$+\frac{1}{2}\left[\widetilde{R}_5\tilde{R}^{k-1}-R_5R^{k-1}\right]R_2\psi+\frac{1}{2}\left[\widetilde{R}_5\tilde{R}^{k-1}-R_5R^{k-1}\right]R_3\tau^2 f_{1,1}$$

$$+\frac{1}{2}[\tilde{R}-R]R_4\tau^2 f_{k-1}+\frac{1}{2}R_4\tau^2\left(\sum_{s=2}^{k-2}\left[R_5R_6R^{k-1-s}-\widetilde{R}_5\widetilde{R}_6\tilde{R}^{k-1-s}\right](f_s-f_{s-1})\right.$$

$$\left.\left[\widehat{R}_5\widehat{R}_6-R_5R_6\right]f_{k-2}-\left[\widehat{R}_5\widehat{R}_6\tilde{R}^{k-2}-R_5R_6R^{k-2}\right]f_1\right)\Big\},2\le k\le N \tag{2.56}$$

elde edilir. Formül (2.56) ya $A^{1/2}$ operatörü ve üçgen eşitsizliği uygulanarak, (2.31) kestirimleri yardımıyla

$$\left\|A^{1/2}\frac{u_k-u_{k-1}}{\tau}\right\|_H$$

$$\le\frac{1}{\tau}\Big\{\frac{1}{2}\Big(\|A^{-1/2}R_5\|_{H\to H}\left\|\widetilde{R}_1\right\|_{H\to H}\|R^{k-1}\|_{H\to H}$$

$$+\left\|A^{-1/2}\widetilde{R}5\right\|_{H\to H}\|R_1\|_{H\to H}\|R^{k-1}\|_{H\to H}\Big)\|A\varphi\|_H$$

$$+\frac{1}{2}\Big(\|A^{1/2}R_2\|_{H\to H}\left\|A^{-1/2}\widetilde{R}5\right\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H}$$

$$+\|A^{1/2}R_2\|_{H\to H}\|A^{-1/2}R_5\|_{H\to H}\|R^{k-1}\|_{H\to H}\Big)\|A^{1/2}\psi\|_H$$

$$+\frac{1}{2}\Big(\|\tau A^{1/2}R_3\|_{H\to H}\left\|A^{-1/2}\widetilde{R}5\right\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H}$$

$$+\|\tau A^{1/2}R_3\|_{H\to H}\|A^{-1/2}R_5\|_{H\to H}\|R^{k-1}\|_{H\to H}\Big)\tau\|A^{1/2}f_{1,1}\|_H$$

$$+\frac{1}{2}\Big(\|\tau A^{1/2}R_4\|_{H\to H}\|\tilde{R}\|_{H\to H}+\|\tau A^{1/2}R_4\|_{H\to H}\|R\|_{H\to H}\Big)\|f_{k-1}\|_H\tau$$

$$+\frac{1}{2}\|\tau A^{1/2}R_4\|_{H\to H}\left(\sum_{s=2}^{k-2}\Big(\|A^{-1/2}R_5\|_{H\to H}\|\tau A^{1/2}R_6\|_{H\to H}\|R^{k-1-s}\|_{H\to H}\right.$$

$$+\left\|A^{-1/2}\widetilde{R}5\right\|_{H\to H}\left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H}\|\tilde{R}^{k-1-s}\|_{H\to H}\Big)\|f_s-f_{s-1}\|_H$$

$$+\left(\|A^{-1/2}\widetilde{R}5\|_{H\to H}\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}+\|A^{-1/2}R_5\|_{H\to H}\|\tau A^{1/2}R_6\|_{H\to H}\right)\|f_{k-2}\|_H$$

$$+\left(\|A^{-1/2}\widetilde{R}5\|_{H\to H}\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}\|\tilde{R}^{k-2}\|_{H\to H}\right.$$

$$\left.+\|A^{-1/2}R_5\|_{H\to H}\|\tau A^{1/2}R_6\|_{H\to H}\|R^{k-2}\|_{H\to H}\right)\|f_1\|_H\Big)\Big\}$$

$$\leq M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\varphi\|_H+\|A^{1/2}\psi\|_H+\tau\|A^{1/2}f_{1,1}\|_H\right\}$$

kararlılık kestirimi elde edildi. Bütün k değerlerinde $\|A^{1/2}\tau^{-1}(u_k-u_{k-1})\|_H$ için elde edilen kestirimler birlikte ele alınırsa

$$\max_{1\leq k\leq N-1}\left\|A^{1/2}\frac{u_k-u_{k-1}}{\tau}\right\|_H\leq M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A^{1/2}\psi\|_H\right.$$

$$\left.+\|A\varphi\|_H+\tau\|A^{1/2}f_{1,1}\|_H\right\} \tag{2.57}$$

kararlılık kestirimi elde edilir. Aynı şekilde, (2.37) ye Abel formülü uygulanarak

$$u_k=\frac{1}{2}\left[\widetilde{R}_1R^k-R_1\tilde{R}^k\right]\varphi+\frac{1}{2}[\tilde{R}^k-R^k]R_2\psi+\frac{1}{2}[\tilde{R}^k-R^k]R_3\tau^2f_{1,1}$$

$$+\frac{1}{2}\tau^2R_4\left(\sum_{s=2}^{k-1}\left[R_6R^{k-s}-\widetilde{R}_6\tilde{R}^{k-s}\right](f_s-f_{s-1})\right.$$

$$\left.+\left(\widehat{R}_6-R_6\right)f_{k-1}-\left[\widehat{R}_6\tilde{R}^{k-1}-R_6R^{k-1}\right]f_1\right),\ 2\leq k\leq N \tag{2.58}$$

formülü elde edilir. Daha sonra (2.58) formülüne A operatörü ve üçgen eşitsizliği uygulanarak, (2.31) kestirimleri yardımı ile, $k\geq 2$ için

$$\|Au_k\|_H\leq\frac{1}{2}\left(\|\widetilde{R}_1\|_{H\to H}\|R^k\|_{H\to H}+\|R_1\|_{H\to H}\|\tilde{R}^k\|_{H\to H}\right)\|A\varphi\|_H$$

$$+\frac{1}{2}\left(\|A^{1/2}R_2\|_{H\to H}\|\tilde{R}^k\|_{H\to H}+\|A^{1/2}R_2\|_{H\to H}\|R^k\|_{H\to H}\right)\|A^{1/2}\psi\|_H$$

$$+\frac{1}{2}\left(\|\tau A^{-1/2}R_3\|_{H\to H}\|\tilde{R}^k\|_{H\to H}+\|\tau A^{-1/2}R_3\|_{H\to H}\|R^k\|_{H\to H}\right)\tau\|A^{1/2}f_{1,1}\|_H$$

$$+\frac{1}{2}\left\|\tau A^{1/2}R_4\right\|_{H\to H}\left(\sum_{s=2}^{k-1}\left[\left\|\tau A^{1/2}R_6\right\|_{H\to H}\left\|R^{k-s}\right\|_{H\to H}+\left\|\tau A^{1/2}\widehat{R}_6\right\|_{H\to H}\left\|\tilde{R}^{k-s}\right\|_{H\to H}\right]\| f_s-f_{s-1}\|_H\right.$$

$$+\left(\left\|\tau A^{1/2}\widehat{R}_6\right\|_{H\to H}+\left\|\tau A^{1/2}R_6\right\|_{H\to H}\right)\| f_{k-1}\|_H$$

$$\left.+\left[\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H}+\|\tau A^{1/2}R_6\|_{H\to H}\|R^{k-1}\|_{H\to H}\right]\| f_1\|_H\right)$$

$$\leq M\left\{\sum_{s=2}^{N-1}\left\|f_s-f_{s-1}\right\|_H+\left\|f_1\right\|_H+\left\|A\varphi\right\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}$$

yazılabilir. Bütün k değerlerinde $\|Au_k\|_H$ için elde edilen kestirimler birlikte değerlendirilirse

$$\max_{1\le k\le N}\left\|Au_k\right\|_H\leq M\sum_{s=2}^{N-1}\left\|f_s-f_{s-1}\right\|_H+\left\|f_1\right\|_H+\left\|A\varphi\right\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H \tag{2.59}$$

kararlılık kestirimi elde edilir.

Formül (2.37) kullanılarak,

$$\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}=\frac{1}{\tau^2}\left\{\frac{1}{2}\left[\widetilde{R}_1R_5^2R^{k-1}-R_1\widetilde{R}_5^2\tilde{R}^{k-1}\right]\varphi\right.$$

$$+\frac{1}{2}\left[\widetilde{R}_5^2\tilde{R}^{k-1}-R_5^2R^{k-1}\right]R_2\psi+\frac{1}{2}\left[\widetilde{R}_5^2\tilde{R}^{k-1}-R_5^2R^{k-1}\right]R_3\tau^2f_{1,1}$$

$$+\frac{1}{2}[\tilde{R}-R]R_4\tau^2f_k+\frac{1}{2}\left(\widetilde{R}_5^2-R_5^2\right)R_4\tau^2f_{k-1}$$

$$+\frac{1}{2}R_4\sum_{s=1}^{k-2}\left[\widehat{R}_5^2\tilde{R}^{k-1-s}-R_5^2R^{k-1-s}\right]f_s\tau^2\Bigg\} \tag{2.60}$$

formülü kolayca elde edilir. İlk olarak (2.60) a Abel formülü uygulanırsa

$$\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} = \frac{1}{\tau^2}\Bigg\{\frac{1}{2}\left[\widetilde{R}_1R_5^2R^{k-1}-R_1\widetilde{R}_5^2\tilde{R}^{k-1}\right]\varphi$$

$$+\frac{1}{2}\left[\widetilde{R}_5^2\tilde{R}^{k-1}-R_5^2R^{k-1}\right]R_2\psi+\frac{1}{2}\left[\widetilde{R}_5^2\tilde{R}^{k-1}-R_5^2R^{k-1}\right]R_3\tau^2f_{1,1}$$

$$+\frac{1}{2}[\tilde{R}-R]R_4\tau^2f_k+\frac{1}{2}\left(\widetilde{R}_5^2-R_5^2\right)R_4\tau^2f_{k-1}$$

$$+\frac{1}{2}R_4\tau^2\Bigg(\sum_{s=2}^{k-2}\left[R_6R_5^2R^{k-1-s}-\widetilde{R}_6\widetilde{R}_5^2\tilde{R}^{k-1-s}\right](f_s-f_{s-1})$$

$$+\left(\widehat{R}_5^2\widehat{R}_6-R_5^2R_6\right)f_{k-2}-\left[\widehat{R}_5^2\widehat{R}_6\tilde{R}^{k-2}-R_5^2R_6R^{k-2}\right]f_1\Bigg)\Bigg\},\ 2\le k\le N \tag{2.61}$$

bulunur. Daha sonra (2.61) formülü ve (2.31) kestirimi kullanılır, üçgen eşitsizliği uygulanırsa

$$\left\|\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\|_H \le \frac{1}{\tau^2}\Bigg\{\frac{1}{2}\Bigg(\left\|\left(A^{-1/2}R_5\right)^2\right\|_{H\to H}\left\|\widehat{R}_1\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}$$

$$+\left\|\left(A^{-1/2}\widehat{R}_5\right)^2\right\|_{H\to H}\left\|\widehat{R}_1\right\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}\Bigg)\|A\varphi\|_H$$

$$+\frac{1}{2}\Bigg(\left\|\left(A^{-1/2}\widehat{R}_5\right)^2\right\|_{H\to H}\left\|A^{1/2}R_2\right\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}$$

$$+\left\|\left(A^{-1/2}R_5\right)^2\right\|_{H\to H}\left\|A^{1/2}R_2\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\Bigg)\left\|A^{1/2}\psi\right\|_H$$

$$+\frac{1}{2}\left(\left\|\left(A^{-1/2}\widehat{R}_5\right)^2\right\|_{H\to H}\left\|\tau A^{1/2}R_3\right\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}+\left\|\left(A^{-1/2}R_5\right)^2\right\|_{H\to H}\left\|\tau A^{1/2}R_3\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right)$$

$$\times\,\tau\|A^{1/2}f_{1,1}\|_H+\frac{1}{2}\left(\|\tau A^{1/2}R_4\|_{H\to H}\|\tilde{R}\|_{H\to H}+\|\tau A^{1/2}R_4\|_{H\to H}\|R\|_{H\to H}\right)\tau\|f_k\|_H$$

$$+\frac{1}{2}\left(\left\|\left(A^{-1/2}\widetilde{R}_5\right)^2\right\|_{H\to H}+\left\|\left(A^{-1/2}\widetilde{R}_5\right)^2\right\|_{H\to H}\right)\|A^{1/2}R_4\|_{H\to H}\tau\|A^{1/2}f_{k-1}\|_H$$

$$+\frac{1}{2}\left\|\tau A^{1/2}R_4\right\|_{H\to H}\left(\sum_{s=2}^{k-2}\left(\left\|\left(A^{-1/2}R_5\right)^2\right\|_{H\to H}\left\|\tau A^{1/2}R_6\right\|_{H\to H}\left\|R^{k-1-s}\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|\left(A^{-1/2}\widetilde{R}_5\right)^2\right\|_{H\to H}\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}\|\widetilde{R}^{k-1-s}\|_{H\to H}\right)\|f_s-f_{s-1}\|_H$$

$$+\left(\left\|\left(A^{-1/2}\widetilde{R}_5\right)^2\right\|_{H\to H}\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}+\left\|(A^{-1/2}R_5)^2\right\|_{H\to H}\|\tau A^{1/2}R_6\|_{H\to H}\right)$$

$$\times\|f_{k-2}\|_H+\left(\left\|\left(A^{-1/2}\widetilde{R}_5\right)^2\right\|_{H\to H}\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}\|\widetilde{R}^{k-2}\|_{H\to H}\right.$$

$$\left.\left.+\left\|(A^{-1/2}R_5)^2\right\|_{H\to H}\|\tau A^{1/2}R_6\|_{H\to H}\|R^{k-2}\|_{H\to H}\right)\|f_1\|_H\right)\Bigg\}$$

$$\le M\left\{\sum_{s=2}^{N-1}\left\|f_s-f_{s-1}\right\|_H+\left\|f_1\right\|_H+\left\|A\varphi\right\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\} \tag{2.62}$$

kararlılık kestirimi elde edilir. Bütün k değerlerinde $\|\tau^{-1}(u_{k+1}-2u_k+u_{k-1})\|_H$ için elde edilen kestirimler birlikte değerlendirilerek

$$\max_{1\le k\le N-1}\left\|\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\|_H\le M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A^{1/2}\psi\|_H\right.$$

$$\left.+\left\|A\varphi\right\|_H+\tau\left\|Af_{1,1}\right\|_H\right\} \tag{2.63}$$

kararlılık kestirimi elde edilir. Kestirimler (2.57), (2.59), (2.63) yardımı ile (2.34) kararlılık kestirimi elde edilir. Böylece Teorem 2.1 in ispatı tamamlanmış olur.

Daha sonra Teorem 2.1 in bazı sonuçları verilecektir. Önce karışık tip

$$\begin{cases} u_{tt} - (a(x)u_x)_x + \delta u = f(t,x),\ 0 < t < 1,\ 0 < x < 1, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), 0 \le x \le 1, \\ u(t,0) = u(t,1),\ u_x(t,0) = u_x(t,1),\ 0 \le t \le 1 \end{cases} \tag{2.64}$$

hiperbolik problemini ele alalım. Problem (2.64) ün bir tek ve düzgün çözümü, $u(t,x)$ ($\delta > 0$) mevcuttur ve $a(x) \ge a > 0$, $(x \in (0,1))$, $\varphi(x), \psi(x)$ $(x \in [0,1])$, $f(t,x)$, $(t, x \in [0,1])$ fonksiyonları düzgün fonksiyonlardır. Bu sayede, Hilbert uzayı $H = L_2[0,1]$ üzerinde, (2.64) teki kendine eşlenik, pozitif tanımlı A^x operatörü yardımı ile karışık tip problem (2.64), Cauchy problem (2.1) e dönüştürülebilir. Problem (2.64) ün ayrışımı (discretization) iki adımda ele alındı. İlk adımda grid uzay $[0,1]_h = \{x \ :\ x_r = rh, 0 \le r \le K, Kh = 1\}$ şeklinde tanımlandı. $[0,1]_h$ aralığında, $\varphi^h(x) = \{\varphi_r\}_1^{K-1}$ grid fonksiyonlarının tanımlı olduğu, Hilbert uzayı $L_{2h} = L_2([0,1]_h)$ ve Sobolev uzayları $W_{2h}^1 = W_{2h}^1([0,1]_h)$, $W_{2h}^2 = W_{2h}^2([0,1]_h)$ verildi. Bunların üzerindeki normlar sırasıyla

$$\| \varphi^h \|_{L_{2h}} = \left(\sum_{r=1}^{K-1} |\varphi^h(x)|^2 h \right)^{1/2},$$

$$\left\| \varphi^h \right\|_{W_{2h}^1} = \left\| \varphi^h \right\|_{L_{2h}} + \left(\sum_{r=1}^{K-1} \left| (\varphi^h)_{\bar{x},j} \right|^2 h \right)^{1/2},$$

$$\| \varphi^h \|_{W_{2h}^2} = \| \varphi^h \|_{L_{2h}} + \left(\sum_{r=1}^{K-1} \left| (\varphi^h)_{\bar{x},j} \right|^2 h \right)^{1/2} + \left(\sum_{r=1}^{K-1} \left| (\varphi^h)_{x\bar{x},j} \right|^2 h \right)^{1/2}$$

şeklinde tanımlanır. Problem (2.64) teki diferansiyel operatörü yerine, $\varphi_0 = \varphi_K$, $\varphi_1 - \varphi_0 = \varphi_K - \varphi_{K-1}$ şartlarını sağlayan, grid fonksiyonlar uzayı $\varphi^h(x) = \{\varphi_r\}_0^K$ da etki eden fark operatörü

$$A_h^x \varphi^h(x) = \left\{ -(a(x)\varphi_{\bar{x}})_{x,r} + \delta\varphi_r \right\}_1^{K-1}, \tag{2.65}$$

i yazalım. A_h^x operatörünün yardımıyla sonsuz çoklukta adi diferansiyel denklem sistemi için

$$\begin{cases} \frac{d^2 v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = f^h(t,x), \ 0 < t < 1, \ x \in (0,1)_h, \\ v^h(0,x) = \varphi^h(x), x \in [0,1]_h, \\ v_t^h(0,x) = \psi^h(x), x \in [0,1]_h \end{cases} \tag{2.66}$$

başlangıç değer problemi elde edilir. İkinci adımda, (2.66) probleminin fark şeması

$$\begin{cases} \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + \frac{2}{3} A_h^x u_k^h(x) + \frac{1}{6} A_h^x (u_{k+1}^h(x) + u_{k-1}^h(x)) \\ + \frac{1}{12}\tau^2 (A_h^x)^2 u_{k+1}^h(x) = f_k^h(x), \\ f_k^h(x) = \frac{2}{3} f^h(t_k, x) + \frac{1}{6}(f^h(t_{k+1}, x) + f^h(t_{k-1}, x)) \\ -\frac{1}{12}\tau^2(-Af^h(t_{k+1}, x) + f_{tt}^h(t_{k+1}, x)), x \in [0,1]_h, \\ t_k = k\tau, N\tau = 1, 1 \le k \le N-1, \\ u_0^h(x) = \varphi^h(x), x \in [0,1]_h, \\ \left(I + \frac{\tau^2}{12}(A_h^x) + \frac{\tau^4}{144}(A_h^x)^2\right)\tau^{-1}(u_1^h(x) - u_0^h(x)) \\ = \left(-(\frac{\tau}{2}(A_h^x))\varphi^h(x) + \left(I - \frac{\tau^2}{12}(A_h^x)\right)\psi^h(x)\right. \\ +\tau f_{1,1}^h(x)), x \in [0,1]_h, \\ f_{1,1}^h(x) = \frac{1}{2} f^h(0,x) + \frac{\tau}{6} f_t^h(0,x). \end{cases} \tag{2.67}$$

yazılır.

Teorem 2. 2 τ ve h yeterince küçük sayılar olsun. Bu takdirde, fark şeması (2.67) nin çözümü için

$$\max_{0 \le k \le N} \| u_k^h \|_{L_{2h}} + \max_{0 \le k \le N} \| u_k^h \|_{W_{2h}^1}$$

$$\le M_1 \left[\max_{1 \le k \le N-1} \| f_k^h \|_{L_{2h}} + \| \psi^h \|_{L_{2h}} + \| \varphi^h \|_{W_{2h}^1} + \tau \| f_{1,1}^h \|_{L_{2h}} \right],$$

$$\max_{1 \le k \le N-1} \| \tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h) \|_{L_{2h}} + \max_{0 \le k \le N} \| u_k^h \|_{W_{2h}^2}$$

$$\leq M_1\left[\left\|f_1^h\right\|_{L_{2h}} + \max_{2\leq k\leq N-1}\left\|\tau^{-1}\left(f_k^h - f_{k-1}^h\right)\right\|_{L_{2h}} + \left\|\psi^h\right\|_{W_{2h}^1} + \left\|\varphi^h\right\|_{W_{2h}^2} + \tau\left\|f_{1,1}^h\right\|_{W_{2h}^1}\right]$$

kararlılık kestirim eşitsizlikleri sağlanır. Burada M_1 sabiti, τ, h, $\varphi^h(x)$, $\psi^h(x)$, $f_{1,1}^h$ ve f_k^h $(1 \leq k < N)$ ifadelerinden bağımsızdır.

Teorem 2.2 nin ispatı soyut (abstract) Teorem 2.1 in ispatına ve (2.65) te verilen A_h^x fark operatörünün simetri özeliğine bağlıdır.

İkinci olarak Ω, m-boyutlu $\mathbb{R}^m\left\{x=(x_1,\cdots,x_m): 0<x_j<1, 1\leq j\leq m\right\}$ Öklid uzayında, sınırları S, $\overline{\Omega}=\Omega\cup S$ olan birim açık küp olsun ve $[0,1]\times\Omega$ de çok boyutlu hiperbolik denklem için karışık sınır değer problemi

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^{m}(a_r(x)u_{x_r})_{x_r} = f(t,x), \\ x=(x_1,\ldots,x_m)\in\Omega,\ 0<t<1, \\ u(0,x)=\varphi(x), u_t(0,x)=\psi(x), \\ x\in\overline{\Omega};\ u(t,x)=0,\ x\in S \end{cases} \tag{2.68}$$

i ele alalım. Burada $a_r(x)$, $(x\in\Omega)$, $\varphi(x)$, $\psi(x)$ $(x\in\overline{\Omega})$, $f(t,x)$ $(t\in(0,1),\ x\in\Omega)$ düzgün fonksiyonlar ve $a_r(x)\geq a>0$ dır.

$\overline{\Omega}$ da karesi integrallenebilen tüm fonksiyonların tanımladığı Hilbert uzayı $L_2(\overline{\Omega})$ de normu $\| f \|_{L_2(\overline{\Omega})}=\{\int\cdots\int_{x\in\overline{\Omega}} | f(x)|^2\, dx_1\cdots dx_m\}^{\frac{1}{2}}$ şeklinde tanımlayalım.

Problem (2.68) in çözümü $u(t,x)$ bir tektir ve düzgün fonksiyondur ayrıca, $\varphi(x)$, $\psi(x)$, $a_r(x)$, $f(t,x)$ fonksiyonları da düzgün fonksiyonlardır. Bu sayede, Hilbert uzayı $H=L_2[0,1]$ üzerinde, (2.68) de kendine eşlenik, pozitif tanımlı bir A^x operatörü kullanılarak karışık tip problem (2.68), Cauchy problem (2.1) e dönüştürülebilir. Problem (2.68) in ayrışımı iki adımda yapılabilir. Birinci adımda grid uzayı

$$\widetilde{\Omega}_h = \{x = x_r = (h_1 r_1, \cdots, h_m r_m), r = (r_1, \cdots, r_m),$$

$$0 \le r_j \le N_j, h_j N_j = 1, j = 1, \cdots, m\}, \Omega_h = \widehat{\Omega}_h \cap \Omega, S_h = \widehat{\Omega}_h \cap S$$

tanımlanır. $\widehat{\Omega}_h$ de $\varphi^h(x) = \{\varphi(h_1 r_1, \cdots, h_m r_m)\}$ grid fonksiyonlarının tanımlı olduğu, Banach uzayı $L_{2h} = L_2(\widehat{\Omega}_h)$ de ve Sobolev uzayı $W_{2h}^1 = W_{2h}^1\left(\widetilde{\Omega}_h\right)$ de, normlar sırasıyla

$$\|\varphi^h\|_{W_{2h}^1} = \|\varphi^h\|_{L_{2h}} + \left(\sum_{x \in \overline{\Omega_h}} \sum_{r=1}^{m} \left| (\varphi^h)_{\overline{x_r}, j_r} \right|^2 h_1 \cdots h_m \right)^{1/2},$$

ve

$$\|\varphi^h\|_{W_{2h}^2} = \|\varphi^h\|_{L_{2h}} + \left(\sum_{x \in \overline{\Omega_h}} \sum_{r=1}^{m} \left| (\varphi^h)_{\overline{x_r}} \right|^2 h_1 \cdots h_m \right)$$

$$+ \left(\sum_{x \in \overline{\Omega_h}} \sum_{r=1}^{m} \left| (\varphi^h)_{x_r \overline{x_r}, j_r} \right|^2 h_1 \cdots h_m \right)^{1/2}$$

şeklinde tanımlanır. Problem (2.68) deki diferansiyel operatörü için, tüm $x \in S_h$ lar için $u^h(x) = 0$ koşullarını sağlayan $u^h(x)$ grid fonksiyonlar uzayında etki eden, A_h^x fark operatörü

$$A_h^x u_x^h = -\sum_{r=1}^{m} \left(a_r(x) u_{\overline{x_r}}^h \right)_{x_r, j_r} \tag{2.69}$$

verildi. Bilindiği üzere A_h^x operatörü $L_2(\widetilde{\Omega}_h)$ de kendine eşlenik ve pozitif tanımlıdır. A_h^x operatörü yardımıyla sonsuz çoklukta adi diferansiyel denklem sistemi için Cauchy problemi

$$\begin{cases} \frac{d^2 v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = f^h(t,x),\ 0 \le t \le 1,\ x \in \Omega_h, \\ v^h(0,x) = \varphi^h(x), x \in \widetilde{\Omega}_h, \\ \frac{dv^h(0,x)}{dt} = \psi^h(x), x \in \widetilde{\Omega}_h \end{cases} \tag{2.70}$$

elde edilir. İkinci adımda problem (2.70) in fark şeması

$$\begin{cases} \frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+\frac{2}{3}A_h^x u_k^h(x)+\frac{1}{6}A_h^x\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right) \\ +\frac{1}{12}\tau^2\left(A_h^x\right)^2 u_{k+1}^h(x)=f_k^h(x), \\ f_k^h(x)=\frac{2}{3}f^h(t_k,x)+\frac{1}{6}\left(f^h(t_{k+1},x)+f^h(t_{k-1},x)\right) \\ -\frac{1}{12}\tau^2\left(-Af^h(t_{k+1},x)+f_{tt}^h(t_{k+1},x)\right),x\in\Omega_h, \\ t_k=k\tau,1\le k\le N-1,N\tau=1, \\ u_0^h(x)=\varphi^h(x),x\in\widehat{\Omega}_h, \\ \left(I+\frac{\tau^2}{12}\left(A_h^x\right)+\frac{\tau^4}{144}\left(A_h^x\right)^2\right)\tau^{-1}\left(u_1^h(x)-u_0^h(x)\right) \\ =\left(-\left(\frac{\tau}{2}\left(A_h^x\right)\right)\varphi^h(x)+\left(I-\frac{\tau^2}{12}\left(A_h^x\right)\right)\psi^h(x)\right. \\ \left.+\tau f_{1,1}^h(x)\right),x\in\widehat{\Omega}_h, \\ f_{1,1}^h(x)=\frac{1}{2}f^h(0,x)+\frac{\tau}{6}f_t^h(0,x) \end{cases} \tag{2.71}$$

oluşturuldu.

Teorem 2.3 τ ve $|h|$ yeterince küçük sayılar olsun. Bu takdirde fark şeması (2.71) in çözümü için aşağıdaki kararlılık kestirimlerini sağlanır;

$$\max_{0\le k\le N}\|u_k^h\|_{L_{2h}}+\max_{0\le k\le N}\|u_k^h\|_{W_{2h}^1}+\max_{1\le k\le N}\|\tau^{-1}(u_k^h-u_{k-1}^h)\|_{L_{2h}}$$

$$\le M_1\left[\max_{1\le k\le N-1}\|f_k^h\|_{L_{2h}}+\|\psi^h\|_{L_{2h}}+\|\varphi^h\|_{W_{2h}^1}+\tau\|f_{1,1}^h\|_{L_{2h}}\right],$$

$$\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}+\max_{0\le k\le N}\left\|u_k^h\right\|_{W_{2h}^2}+\max_{1\le k\le N}\left\|\tau^{-1}\left(u_k^h-u_{k-1}^h\right)\right\|_{W_{2h}^1}$$

$$\le M_1\left[\left\|f_1^h\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\tau^{-1}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}+\left\|\psi^h\right\|_{W_{2h}^1}+\left\|\varphi^h\right\|_{W_{2h}^2}+\tau\left\|f_{1,1}^h\right\|_{W_{2h}^1}\right].$$

Burada M_1 sabiti $\tau,\ h,\ \varphi^h(x),$ $\psi^h(x), f_{1,1}^h$ ve f_k^h $(1\le k<N)$ ifadelerinden

bağımsızdır.

Teorem 2.3 ün ispatı Teorem 2.1 in ispatına, (2.69) da tanımlanan A_h^x operatörünün simetri özeliğine ve aşağıda verilmiş olan L_{2h} da eliptik fark problemlerinde coercivity eşitsizliğiyle ilgili teoreme bağlıdır.

Teorem 2.4 Eliptik fark problemi

$$A_h^x u^h(x) = \omega^h(x), x \in \Omega_h,$$
$$u^h(x) = 0, x \in S_h$$

in çözümü için

$$\sum_{r=1}^{m} \left\| u^h_{x_r \bar{x}_r, j_r} \right\|_{L_{2h}} \leq M \| \omega^h \|_{L_{2h}}$$

eşitsizliği sağlanır. Burada M sabiti, h ve $\omega^h(x)$ ifadelerinden bağımsızdır (Sobolevskii 1975).

2.3 Hiperbolik Tip Cauchy Problemi için Dördüncü Mertebeden Doğruluk Fark Şeması

Bu kısımda, hiperbolik tip Cauchy problemi için, dördüncü mertebeden kararlı fark şeması elde alınacak. (2.7) ve (2.20) formülleri kullanılarak, Cauchy problem (2.1) in yaklaşık çözümü için, dördüncü mertebeden doğruluk fark şeması

$$
\begin{cases}
\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{5}{6}Au_k+\frac{1}{12}A\left(u_{k+1}+u_{k-1}\right)\\
-\frac{\tau^2}{72}A^2u_k+\frac{\tau^2}{144}A^2\left(u_{k+1}+u_{k-1}\right)=f_k,\\
f_k=\frac{5}{6}f(t_k)+\frac{1}{12}\left(f\left(t_{k+1}\right)+f\left(t_{k-1}\right)\right)+\frac{\tau^2}{72}\left(-Af(t_k)+f''(t_k)\right)\\
-\frac{1}{144}\tau^2\left(-A\left(f(t_{k+1})+f(t_{k-1})\right)+f''(t_{k+1})+f''(t_{k-1})\right),\\
1\le k\le N-1, t_k=k\tau, N\tau=1,\\
u_0=\varphi,\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\tau^{-1}\left(u_1-u_0\right)\\
=-\frac{\tau}{2}A\varphi+\left(I-\frac{\tau^2A}{12}\right)\psi+f_{2,2}\tau,
\end{cases} \tag{2.72}
$$

elde edildi. Burada

$$
f_{2,2}=\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\tau^{-1}f_{1,1}^4
$$

$$
=\left\{\left(I-\frac{\tau^2A}{12}\right)f(0)+\left(-\left(I-\frac{5\tau^2A}{12}\right)f(0)+\tau f'(0)\right)\frac{1}{2}\right.
$$

$$
\left.+\left(-A\tau f(0)-2f'(0)+\tau f''(0)\right)\frac{\tau}{6}+\left(Af(0)-3f''(0)\right)\frac{\tau^2}{24}\right\}
$$

tür. Bu kısımda ayrıca, formül (2.12) de tanımlanan R, $\tilde{R}$ ve

$$
J_1=\left(I-\frac{\tau^2A}{12}\right)\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^{-1},\ \widehat{J}_1=\left(I-\frac{\tau^2A}{12}\right)\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^{-1},
$$

$$
J_2=\left(-i\tau A^{1/2}\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)\right)^{-1},\ \widehat{J}_2=\left(-i\tau A^{1/2}\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)\right)^{-1},
$$

$$
J_3=i\tau A^{1/2}\left(I-\frac{\tau^2A}{12}\right)\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^{-2},
$$

$$
\widehat{J}_3=-i\tau A^{1/2}\left(I-\frac{\tau^2A}{12}\right)\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^{-2},
$$

$$J_4=\left(I-\frac{\tau^2A}{12}\right)\left(i\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\right)^{-1},$$

$$\widehat{J}_4=\left(I-\frac{\tau^2A}{12}\right)\left(-i\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\right)^{-1}$$

operatörlerinden yararlanıldı. Burada J_1, $\widehat{J}_1$, J_2, $\widehat{J}_2$, J_3, $\widehat{J}_3$ ve J_4, $\widehat{J}_4$ den aynı alt indisliler eşleniktir.

Öncelikle bu kısımdaki teoremlerin ispat aşamalarında yararlanılacak olan bir lemma ele alalım.

Lemma 2.2 Aşağıdaki kestirimler sağlanır.

$$\begin{cases} \|R\|_{H\to H}\le 1, \quad \|\tilde{R}\|_{H\to H}\le 1, \\ \|J_1\|_{H\to H}\le 1, \quad \|\widehat{J}_1\|_{H\to H}\le 1, \\ \|\tau A^{1/2}J_2\|_{H\to H}\le 1, \quad \|\tau A^{1/2}\widehat{J}_2\|_{H\to H}\le 1, \\ \|A^{-1/2}J_3\|_{H\to H}\le \tau, \quad \|A^{-1/2}\widehat{J}_3\|_{H\to H}\le \tau, \\ \|\tau A^{1/2}J_4\|_{H\to H}\le 1, \quad \|\tau A^{1/2}\widehat{J}_4\|_{H\to H}\le 1. \end{cases} \tag{2.73}$$

İspat: Kendisine eşlenik, pozitif tanımlı operatörün spektral özeliği yardımıyla

$$\left\|\left(I\pm\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)\left(I\mp\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^{-1}\right\|_{H\to H}\le \underset{\delta\le\lambda<\infty}{Sup}\left|\frac{I\pm\frac{1}{2}i\tau\lambda^{1/2}-\frac{1}{12}\tau^2\lambda}{I\mp\frac{1}{2}i\tau\lambda^{1/2}-\frac{1}{12}\tau^2\lambda}\right|$$

yazılabilir. Buradan

$$\frac{\left(1-\frac{1}{12}\tau^2\lambda\right)^2+\frac{1}{4}\tau^2\lambda}{\left(1-\frac{1}{12}\tau^2\lambda\right)^2+\frac{1}{4}\tau^2\lambda}\leq 1$$

olduğundan, $\|R\|_{H\to H}\leq 1,\ \|\tilde{R}\|_{H\to H}\leq 1$ dir. Benzer şekilde

$$\left\|\left(I-\frac{\tau^2A}{12}\right)\left(I\mp\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^{-1}\right\|_{H\to H}\leq \sup_{\delta\leq\lambda<\infty}\left|\frac{1-\frac{\tau^2\lambda}{12}}{1\mp\frac{i\tau\lambda^{1/2}}{2}-\frac{\tau^2\lambda}{12}}\right|$$

yazılabilir. Buradan

$$\frac{\left(1-\frac{\tau^2\lambda}{12}\right)^2}{\left(1-\frac{\tau^2\lambda}{12}\right)^2+\frac{\tau^2\lambda}{4}}=\frac{1-\frac{\tau^2\lambda}{6}+\frac{\tau^4\lambda^2}{144}}{1+\frac{\tau^2\lambda}{12}+\frac{\tau^4\lambda^2}{144}}\leq 1$$

olduğundan, $\|J_1\|_{H\to H}\leq 1,\ \|\widehat{J}_1\|_{H\to H}\leq 1$ dir. Aynı yolla

$$\left\|\tau A^{1/2}\left(i\tau A^{1/2}\left(I\mp\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)\right)^{-1}\right\|_{H\to H}\leq \sup_{\delta\leq\lambda<\infty}\left|\frac{i}{1\mp\frac{i\tau\lambda^{1/2}}{2}-\frac{\tau^2\lambda}{12}}\right|$$

dir. Buradan

$$\frac{1}{\left(1-\frac{\tau^2\lambda}{12}\right)^2+\frac{\tau^2\lambda}{4}}=\frac{1}{1+\frac{\tau^2\lambda}{12}+\frac{\tau^4\lambda^2}{144}}\leq 1$$

olduğundan, $\|\tau A^{1/2}J_2\|_{H\to H}\leq 1,\ \|\tau A^{1/2}\widehat{J}_2\|_{H\to H}\leq 1$ dir. Aynı yöntemle

$$\left\|A^{-1/2}i\tau A^{1/2}\left(I-\frac{\tau^2A}{12}\right)\left(I\mp\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^{-2}\right\|_{H\to H}\leq \sup_{\delta\leq\lambda<\infty}\left|\frac{i\left(1-\frac{\tau^2\lambda}{12}\right)\tau}{\left(1\mp\frac{i\tau\lambda^{1/2}}{2}-\frac{\tau^2\lambda}{12}\right)^2}\right|$$

dir. Buradan

$$\frac{\left(1-\frac{\tau^2\lambda}{12}\right)^2\tau^2}{\left(1-\frac{5\tau^2\lambda}{12}+\frac{\tau^4\lambda^2}{144}\right)^2+\tau^2\lambda\left(1-\frac{\tau^2\lambda}{12}\right)^2}\leq\tau^2$$

olduğundan, $\left\|A^{-1/2}J_3\right\|_{H\to H}\leq\tau$, $\left\|A^{-1/2}\widehat{J}_3\right\|_{H\to H}\leq\tau$ dur. Aynı şekilde

$$\left\|\tau A^{1/2}\left(I-\frac{\tau^2A}{12}\right)\left(\pm i\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\right)^{-1}\right\|_{H\to H}\leq\underset{\delta\leq\lambda<\infty}{Sup}\left|\frac{i\left(1-\frac{\tau^2\lambda}{12}\right)}{1+\frac{\tau^2\lambda}{12}+\frac{\tau^4\lambda^2}{144}}\right|$$

dir. Buradan

$$\left|\frac{\left(1-\frac{\tau^2\lambda}{12}\right)^2}{\left(1+\frac{\tau^2\lambda}{12}+\frac{\tau^4\lambda^2}{144}\right)^2}\right|\leq 1$$

olduğundan, $\left\|\tau A^{1/2}J_4\right\|_{H\to H}\leq 1$, $\left\|\tau A^{1/2}\widehat{J}_4\right\|_{H\to H}\leq 1$ dir. Bu Lemma 2.2 yi ispat eder.

Teorem 2.5 $\varphi\in D(A)$, $\psi\in D(A^{\frac{1}{2}})$, $f_{2,2}\in D(A^{\frac{1}{2}})$ olsun. Bu takdirde fark şeması (2.72) nin çözümü için aşağıdaki kararlılık kestirimleri

$$\max_{1\leq k\leq N}\left\|\frac{u_k+u_{k-1}}{2}\right\|_H\leq M\left\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\|\varphi\|_H\right.$$

$$\left.+\|A^{-1/2}\psi\|_H+\tau\|A^{-1/2}f_{2,2}\|_H\right\}, \tag{2.74}$$

$$\max_{1\leq k\leq N-1}\left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H+\max_{1\leq k\leq N}\left\|A^{1/2}\frac{u_k+u_{k-1}}{2}\right\|_H$$

$$\leq M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|A^{1/2}\varphi\|_H+\|\psi\|_H+\tau\|f_{2,2}\|_H\right\}, \tag{2.75}$$

$$\max_{1\leq k\leq N-1}\left\|A^{1/2}\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H+\max_{1\leq k\leq N}\left\|A\frac{u_k+u_{k-1}}{2}\right\|_H$$

$$\leq M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\varphi\|_H+\|A^{1/2}\psi\|_H+\tau\|A^{1/2}f_{2,2}\|_H\right\}, \tag{2.76}$$

sağlanır. Burada M sabiti, τ, φ, ψ, $f_{2,2}$ ve f_s $(1\leq s\leq N-1)$ ifadelerinden bağımsızdır.

İspat: İlk olarak Cauchy problem (2.1) in dördüncü mertebeden yaklaşık çözümü için formül elde edelim. Teorem 2.1 de uygulanan adımların aynısı takip edilerek başlangıç değer problemi (2.35) in çözümü için (2.36) formülü elde edidi. Problem (2.35) yeniden yazılarak fark denklemi

$$\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)u_{k-1}-\left(2-\frac{5\tau^2A}{6}+\frac{\tau^4A^2}{72}\right)u_k$$

$$+\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)u_{k+1}=\tau^2f_k,\ 1\leq k\leq N-1,$$

$$u_0=\varphi,u_1=\varphi+\tau\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\left(-\frac{\tau}{2}A\varphi+\left(I-\frac{\tau^2A}{12}\right)\psi+\tau f_{2,2}\right) \tag{2.77}$$

elde edilir. Denklemde, sırasıyla a ile $\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)$, b ile $\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)$,

c ile $\left(2-\frac{5\tau^2A}{6}+\frac{\tau^4A^2}{72}\right)$, ψ ile

$$\varphi+\tau\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\left(-\frac{\tau}{2}A\varphi+\left(I-\frac{\tau^2 A}{12}\right)\psi+\tau f_{2,2}\right)$$

ve φ_k ile $\tau^2 f_k$ değiştirilerek ve (2.36) formülü uygulanarak, u_k $(2 \le k \le N)$ için

$$u_k = \frac{1}{2}[R^k+\tilde{R}^k]\varphi+\frac{1}{2}[\tilde{R}^k-R^k]iA^{-1/2}\psi$$

$$+\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\left(\widetilde{R}-R\right)^{-1}[\tilde{R}^k-R^k]\tau^2 f_{2,2}$$

$$+\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\left(\widehat{R}-R\right)^{-1}\sum_{s=1}^{k-1}\left[\tilde{R}^{k-s}-R^{k-s}\right]\tau^2 f_s \qquad (2.78)$$

formülü elde edilir. Şimdi (2.74), (2.75), (2.76) kararlılık kestirimlerini ispatlayalım. Öncelikle, fark şeması (2.72) den yararlanarak

$$\frac{u_1+u_0}{2} = \left(I-\frac{\tau^2 A}{6}+\frac{\tau^4 A^2}{144}\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\varphi$$

$$+\frac{1}{2}\left(I-\frac{\tau^2 A}{12}\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\tau\psi+\frac{1}{2}\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\tau^2 f_{2,2} \qquad (2.79)$$

elde edilir. Daha sonra (2.73) kestirimleri, $\frac{u_1+u_0}{2}$ formülü ve

$$\left\|\left(I-\frac{\tau^2 A}{6}+\frac{\tau^4 A^2}{144}\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H}\le 1, \qquad (2.80)$$

$$\left\|\tau A^{1/2}\left(I-\frac{\tau^2 A}{12}\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H}\le 12, \qquad (2.81)$$

$$\left\| \tau A^{1/2}\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H} \le \frac{12\sqrt{11}}{12+\sqrt{11}}, \tag{2.82}$$

basit kestirimleri kullanılarak

$$\left\| \frac{u_1+u_0}{2} \right\|_H \le \left\| \left(I-\frac{\tau^2 A}{6}+\frac{\tau^4 A^2}{144}\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \|\varphi\|_H$$

$$+\frac{1}{2}\left\| \tau A^{1/2}\left(I-\frac{\tau^2 A}{12}\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \|A^{-1/2}\psi\|_H$$

$$+\frac{1}{2}\left\| \tau A^{1/2}\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \tau \|A^{-1/2} f_{2,2}\|_H$$

$$\le \|\varphi\|_H + 6\|A^{-1/2}\psi\|_H + \frac{6\sqrt{11}}{12+\sqrt{11}}\tau\|A^{-1/2} f_{2,2}\|_H$$

$$\le M\left\{ \sum_{s=1}^{N-1} \left\|A^{-1/2} f_s\right\|_H \tau + \|\varphi\|_H + \left\|A^{-1/2}\psi\right\|_H + \tau\left\|A^{-1/2} f_{2,2}\right\|_H \right\} \tag{2.82a}$$

kararlılık kestirimi elde edilir. $\frac{u_1+u_0}{2}$ formülüne $A^{1/2}$ operatörü uygulanarak ve (2.73), (2.80), (2.81).kestirimlerinden yararlanarak

$$\left\| A^{1/2}\frac{u_1+u_0}{2} \right\|_H \le \left\| \left(I-\frac{\tau^2 A}{6}+\frac{\tau^4 A^2}{144}\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \|A^{1/2}\varphi\|_H$$

$$+\frac{1}{2}\left\| \tau A^{1/2}\left(I-\frac{\tau^2 A}{12}\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \|\psi\|_H$$

$$+\frac{1}{2}\left\|\tau A^{1/2}\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|f_{2,2}\|_H \le \|A^{1/2}\varphi\|_H+6\|\psi\|_H$$

$$+\frac{6\sqrt{11}}{12+\sqrt{11}}\tau\left\|f_{2,2}\right\|_H \le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\,\tau+\left\|A^{1/2}\varphi\right\|_H+\|\psi\|_H+\tau\left\|f_{2,2}\right\|_H\right\}$$

kararlılık kestirimi bulunur. Benzer şekilde, $\frac{u_1+u_0}{2}$ formülüne A operatörü uygulanarak ve (2.73), (2.80), (2.81), (2.82) kestirimleri yardımı ile

$$\left\|A\frac{u_1+u_0}{2}\right\|_H \le \left\|\left(I-\frac{\tau^2 A}{6}+\frac{\tau^4 A^2}{144}\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H}\|A\varphi\|_H$$

$$+\frac{1}{2}\left\|\tau A^{1/2}\left(I-\frac{\tau^2 A}{12}\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H}\|A^{1/2}\psi\|_H$$

$$+\frac{1}{2}\left\|\tau A^{1/2}\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|A^{1/2}f_{2,2}\|_H \le \|A\varphi\|_H+6\|A^{1/2}\psi\|_H$$

$$+\frac{6\sqrt{11}}{12+\sqrt{11}}\tau\|A^{1/2}f_{2,2}\|_H$$

$$\le M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\varphi\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{2,2}\right\|_H\right\} \qquad (2.83)$$

kararlılık kestirimi elde edilir. Fark şeması (2.72) formülü kullanılarak

$$\frac{u_2-u_0}{2\tau}=-\frac{\tau A}{2}\left(I-\frac{\tau^2 A}{6}+\frac{\tau^4 A^2}{144}\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-2}\varphi$$

$$+\left(I-\frac{5\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)\left(I-\frac{\tau^2}{12}A\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-2}\psi$$

$$+\left(I-\frac{5\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-2}\tau f_{2,2}$$

$$+\frac{1}{2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau f_1 \tag{2.84}$$

formülü elde edilir. Daha sonra (2.84) formülünden, (2.73), (2.80), (2.82) ve

$$\left\|\left(I-\frac{5\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\leq 1, \tag{2.85}$$

$$\left\|\left(I-\frac{\tau^2}{12}A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\leq 1, \tag{2.86}$$

$$\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\leq 1, \tag{2.87}$$

basit kestirimlerinden yararlanarak

$$\left\|\frac{u_2-u_0}{2\tau}\right\|_H\leq\left\|\left(I-\frac{\tau^2A}{6}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)^{-1}\right\|$$

$$\times\left\|\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\|A^{1/2}\varphi\|_H$$

$$+\left\|\left(I-\frac{5\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}$$

$$\times\left\|\left(I-\frac{\tau^2}{12}A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\|\psi\|_H$$

$$+\left\|\left(I-\frac{5\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}$$

$$\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\left\|f_{2,2}\right\|_H+\frac{1}{2}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\left\|f_1\right\|$$

$$\leq \frac{12\sqrt{11}}{12+\sqrt{11}}\left\|A^{1/2}\varphi\right\|_H+\left\|\psi\right\|_H+\tau\left\|f_{2,2}\right\|_H+\frac{1}{2}\tau\left\|f_1\right\|_H$$

$$\leq M\left\{\sum_{s=1}^{N-1}\left\|f_s\right\|_H\tau+\left\|f_1\right\|\tau+\left\|A^{1/2}\varphi\right\|_H+\left\|\psi\right\|_H+\tau\left\|f_{2,2}\right\|_H\right\} \tag{2.88}$$

kararlılık kestirimi elde edilir. Benzer şekilde (2.84) formülüne, $A^{1/2}$ operatörü uygulanarak ve (2.73), (2.82), (2.86), (2.87) kestirimlerinden yararlanarak

$$\left\|A^{1/2}\frac{u_2-u_0}{2\tau}\right\|_H \leq \left\|\left(I-\frac{\tau^2A}{6}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|$$

$$\times\left\|\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|A\varphi\right\|_H$$

$$+\left\|\left(I-\frac{5\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}$$

$$\times\left\|\left(I-\frac{\tau^2}{12}A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|A^{1/2}\psi\right\|_H$$

$$+\left\|\left(I-\frac{5\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}$$

$$+\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{1/2}f_{2,2}\right\|_H+\frac{1}{2}\left\|\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|f_1\right\|_H$$

$$\leq \frac{12\sqrt{11}}{12+\sqrt{11}} \|A\varphi\|_H + \frac{6\sqrt{11}}{12+\sqrt{11}} \|f_1\|_H + \|A^{1/2}\psi\|_H + \tau\|A^{1/2}f_{2,2}\|_H$$

$$\leq M \left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A\varphi\|_H + \left\|A^{1/2}\psi\right\|_H + \tau\left\|A^{1/2} f_{2,2}\right\|_H \right\} \tag{2.89}$$

kararlılık kestirimi elde edilir. Sonuç olarak $k = 1$ hali için

$$\left\| \frac{u_k + u_{k-1}}{2} \right\|_H \leq M \left\{ \sum_{s=1}^{N-1} \|A^{-1/2} f_s\|_H \tau + \|\varphi\|_H + \|A^{-1/2}\psi\|_H + \tau\|A^{-1/2} f_{2,2}\|_H \right\},$$

$$\left\| A^{1/2} \frac{u_k + u_{k-1}}{2} \right\|_H \leq M \left\{ \sum_{s=1}^{N-1} \|f_s\|_H \tau + \|A^{1/2}\varphi\|_H + \|\psi\|_H + \tau\|f_{2,2}\|_H \right\},$$

$$\left\| A \frac{u_k + u_{k-1}}{2} \right\|_H \leq M \left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A\varphi\|_H \right.$$

$$\left. + \|A^{1/2}\psi\|_H + \tau\|A^{1/2} f_{2,2}\|_H \right\},$$

$$\left\| \frac{u_{k+1} - u_{k-1}}{2\tau} \right\|_H \leq M \left\{ \sum_{s=1}^{N-1} \|f_s\|_H \tau + \|A^{1/2}\varphi\|_H + \|\psi\|_H + \tau\|f_{2,2}\|_H \right\},$$

$$\left\| A^{1/2} \frac{u_{k+1} - u_{k-1}}{2\tau} \right\|_H \leq M \left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A\varphi\|_H \right.$$

$$\left. + \|A^{1/2}\psi\|_H + \tau\|A^{1/2} f_{2,2}\|_H \right\},$$

kararlılık kestirimleri ispat edildi. Şimdi, tüm $k \geq 2$ değerleri için ispat edelim. Öncelikle (2.86) formülü kullanılarak

$$\frac{u_k + u_{k-1}}{2} = \frac{1}{2}\left[J_1 R^{k-1} + \widetilde{J}_1 \tilde{R}^{k-1}\right]\varphi + \frac{1}{2}\left[\widetilde{J}_1 \tilde{R}^{k-1} - J_1 R^{k-1}\right] i A^{-1/2}\psi$$

$$+ \frac{1}{2}\left[\widetilde{J}_2 \tilde{R}^{k-1} - J_2 R^{k-1}\right]\tau^2 f_{2,2} + \frac{1}{2}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1}\tau^2 f_{k-1}$$

$$+\frac{1}{2}\sum_{s=1}^{k-2}\left[\widehat{J}_2 \tilde{R}^{k-1-s} - J_2 R^{k-1-s}\right]\tau^2 f_s \qquad (2.90)$$

elde edilir. Formül (2.90) dan ve (2.73) kestirimlerinden yararlanarak, üçgen eşitsizliği uygulanırsa, $k \geq 2$ hali için

$$\left\| \frac{u_k + u_{k-1}}{2} \right\|_H \leq \frac{1}{2}\left(\|J_1\|_{H\to H}\|R^{k-1}\|_{H\to H} + \left\|\widetilde{J}_1\right\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H}\right)\|\varphi\|_H$$

$$+ \frac{1}{2}|i|\left(\left\|\widetilde{J}_1\right\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H} + \|J_1\|_{H\to H}\|R^{k-1}\|_{H\to H}\right)\|A^{-1/2}\psi\|_H$$

$$+ \frac{1}{2}\left(\left\|\tau A^{1/2}\widetilde{J}_2\right\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H} + \|\tau A^{1/2} J_2\|_{H\to H}\|R^{k-1}\|_{H\to H}\right)\tau\|A^{-1/2} f_{2,2}\|_H$$

$$+ \frac{1}{2}\left\|\tau A^{1/2}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|A^{-1/2} f_{k-1}\|_H$$

$$+ \frac{1}{2}\sum_{s=1}^{k-2}\left[\left\|\tau A^{1/2}\widetilde{J}_2\right\|_{H\to H}\|\tilde{R}^{k-1-s}\|_{H\to H} + \|\tau A^{1/2} J_2\|_{H\to H}\|R^{k-1-s}\|_{H\to H}\right]\|A^{-1/2} f_s\|_H\tau$$

$$\leq M\left\{\sum_{s=1}^{N-1}\|A^{-1/2} f_s\|_H\tau + \|\varphi\|_H + \|A^{-1/2}\psi\|_H + \tau\|A^{-1/2} f_{2,2}\|_H\right\} \qquad (2.90a)$$

kararlılık kestirimi elde edilir. Bütün k değerlerinde $\left\| \frac{u_k + u_{k-1}}{2} \right\|_H$ için kestirimler birlikte değerlendirilirse (2.74) kararlılık kestirimi elde edilir.

Formül (2.90) a $A^{1/2}$ operatörü ve üçgen eşitsizliği uygulanır, (2.73) kestirimi kullanılırsa, $k \geq 2$ hali için

$$\left\| A^{1/2} \frac{u_k + u_{k-1}}{2} \right\|_H \leq \frac{1}{2}\left(\| J_1 \|_{H \to H} \| R^{k-1} \|_{H \to H} + \| \tilde{J}_1 \|_{H \to H} \| \tilde{R}^{k-1} \|_{H \to H} \right)$$

$$\times \| A^{1/2} \varphi \|_H + \frac{1}{2} |i| \left(\| \tilde{J}_1 \|_{H \to H} \| \tilde{R}^{k-1} \|_{H \to H} + \| J_1 \|_{H \to H} \| R^{k-1} \|_{H \to H} \right) \| \psi \|_H$$

$$+ \frac{1}{2}\left(\| \tau A^{1/2} \tilde{J}_2 \|_{H \to H} \| \tilde{R}^{k-1} \|_{H \to H} + \| \tau A^{1/2} J_2 \|_{H \to H} \| R^{k-1} \|_{H \to H} \right) \tau \| f_{2,2} \|_H$$

$$+ \frac{1}{2} \left\| \tau A^{1/2} \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-1} \right\|_{H \to H} \tau \| f_{k-1} \|_H$$

$$+ \frac{1}{2} \sum_{s=1}^{k-2} \left[\| \tau A^{1/2} \tilde{J}_2 \|_{H \to H} \| \tilde{R}^{k-1-s} \|_{H \to H} + \| \tau A^{1/2} J_2 \|_{H \to H} \| R^{k-1-s} \|_{H \to H} \right] \| f_s \|_H \tau$$

$$\leq M \left\{ \sum_{s=1}^{N-1} \| f_s \|_H \tau + \| A^{1/2} \varphi \|_H + \| \psi \|_H + \tau \| f_{2,2} \|_H \right\} \tag{2.90b}$$

kararlılık kestirimi elde edilir. Bütün k değerlerinde $\left\| A^{1/2} \frac{u_k + u_{k-1}}{2} \right\|_H$ için kestirimler birlikte değerlendirilerek,

$$\max_{1 \leq k \leq N} \left\| A^{1/2} \frac{u_k + u_{k-1}}{2} \right\|_H \leq M \left\{ \sum_{s=1}^{N-1} \| f \|_{sH} \tau + \| A^{1/2} \varphi \|_H + \| \psi \|_H + \tau \left\| f_{2,2} \right\|_H \right\} \tag{2.91}$$

elde edilir.

(2.90) a Abel formülü uygulanarak

$$\frac{u_k + u_{k-1}}{2} = \frac{1}{2}\left[J_1 R^{k-1} + \tilde{J}_1 \tilde{R}^{k-1}\right]\varphi + \frac{1}{2}\left[\tilde{J}_1 \tilde{R}^{k-1} - J_1 R^{k-1}\right] i A^{-1/2}\psi$$

$$+ \frac{1}{2}\left[\tilde{J}_2 \tilde{R}^{k-1} - J_2 R^{k-1}\right]\tau^2 f_{2,2} + \frac{1}{2}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1}\tau^2 f_{k-1}$$

$$+\frac{1}{2}A^{-1}\left(-\sum_{s=2}^{k-2}\left[\tilde{R}^{k-s} + R^{k-s}\right](f_s - f_{s-1}) + +\left(\widehat{R} + R\right) f_{k-2} - \left(\tilde{R}^{k-1} + R^{k-1}\right) f_1\right) \tag{2.92}$$

formülü kolayca yazılabilir. Daha sonra, (2.92) ye A operatörü ve üçgen eşitsizliği uygulanır, (2.73) kestirimleri kullanılırsa

$$\left\| A\frac{u_k + u_{k-1}}{2} \right\|_H \leq \frac{1}{2}\left(\|J_1\|_{H\to H}\|R^{k-1}\|_{H\to H} + \left\|\tilde{J}_1\right\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H}\right)$$

$$\times \|A\varphi\|_H + \frac{1}{2}|i|\left(\left\|\tilde{J}_1\right\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H} + \|J_1\|_{H\to H}\|R^{k-1}\|_{H\to H}\right)\|A^{1/2}\psi\|_H$$

$$+ \frac{1}{2}\left(\left\|\tau A^{1/2}\tilde{J}_2\right\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H} + \|\tau A^{1/2} J_2\|_{H\to H}\|R^{k-1}\|_{H\to H}\right)\tau\|A^{1/2} f_{2,2}\|_H$$

$$+ \frac{1}{2}\left\|\tau A^{1/2}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|A^{1/2} f_{k-1}\|_H$$

$$+ \frac{1}{2}\left(\sum_{s=2}^{k-2}\left(\|\tilde{R}^{k-s}\|_{H\to H} + \|R^{k-s}\|_{H\to H}\right)\|f_s - f_{s-1}\|_H +\right.$$

$$\left. +\left(\|\tilde{R}\|_{H\to H} + \|R\|_{H\to H}\right)\|f_{k-2}\|_H + \left(\|\tilde{R}^{k-1}\|_{H\to H} + \|R^{k-1}\|_{H\to H}\right)\|f_1\|_H\right)$$

$$\leq M\left\{\sum_{s=2}^{N-1}\left\|f_s - f_{s-1}\right\|_H + \left\|f_1\right\|_H + \left\|A\varphi\right\|_H + \left\|A^{1/2}\psi\right\|_H + \tau\left\|A^{1/2} f_{2,2}\right\|_H\right\} \tag{2.92a}$$

kararlılık kestirimi elde edilir. Bütün k değerlinde $\left\| A\frac{u_k+u_{k-1}}{2} \right\|_H$ için kestirimler birlikte değerlendirilerek

$$\max_{1\le k\le N-1} \left\| A\frac{u_k+u_{k-1}}{2} \right\|_H \le M\left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A^{1/2}\psi\|_H \right.$$

$$\left. + \|A\varphi\|_H + \tau \left\|A^{1/2} f_{2,2}\right\|_H \right\} \qquad (2.93)$$

kararlılık kestirimi elde edilir. Benzer şekilde, (2.78) formülü yardımı ile

$$\frac{u_{k+1}-u_{k-1}}{2\tau} = \frac{1}{2\tau}\left[J_3R^{k-1} + \tilde{J}_3\tilde{R}^{k-1}\right]\varphi$$

$$+ \frac{1}{2\tau}\left[\tilde{J}_3\tilde{R}^{k-1} - J_3R^{k-1}\right]iA^{-1/2}\psi + \frac{1}{2\tau}[\tilde{R}^k + R^k]\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4A^2}{144}\right)^{-1}\tau^2 f_{2,2}$$

$$+ \frac{1}{2\tau}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4A^2}{144}\right)^{-1}\tau^2 f_k + \frac{1}{2\tau}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4A^2}{144}\right)^{-1}[\tilde{R}+R]\tau^2 f_{k-1}$$

$$+\frac{1}{2\tau}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4A^2}{144}\right)^{-1} \sum_{s=1}^{k-2}\left[\tilde{R}^{k-s} + R^{k-s}\right] f_s\tau^2 \qquad (2.94)$$

bulunur. Daha sonra (2.94) formülü ve (2.73) kestirimleri kullanılır, üçgen eşitsizliği uygulanırsa

$$\left\| \frac{u_{k+1}-u_{k-1}}{2\tau} \right\|_H$$

$$\le \frac{1}{2\tau}\left(\|A^{-1/2}J_3\|_{H\to H}\|R^{k-1}\|_{H\to H} + \left\|A^{-1/2}\tilde{J}_3\right\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}\right)\|A^{1/2}\varphi\|_H$$

$$+ \frac{1}{2}|i|\left(\left\|A^{-1/2}\tilde{J}_3\right\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H} + \|A^{-1/2}J_3\|_{H\to H}\|R^{k-1}\|_{H\to H}\right)\|\psi\|_H$$

$$+\frac{1}{2}\left(\|\tilde{R}^k\|_{H\to H}+\|R^k\|_{H\to H}\right)\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\|f_{2,2}\|_H\tau$$

$$+\frac{1}{2}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\|f_k\|_H\tau$$

$$+\frac{1}{2}\left(\|\tilde{R}\|_{H\to H}+\|R\|_{H\to H}\right)\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\|f_{k-1}\|_H\tau$$

$$+\frac{1}{2}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\sum_{s=1}^{k-2}\left(\left\|\tilde{R}^{k-s}\right\|_{H\to H}+\left\|R^{k-s}\right\|_{H\to H}\right)\left\|f_s\right\|_H\tau$$

$$\le M\left\{\sum_{s=1}^{N-1}\left\|f_s\right\|_H\tau+\left\|A^{1/2}\varphi\right\|_H+\left\|\psi\right\|_H+\tau\left\|f_{2,2}\right\|_H\right\} \tag{2.95}$$

kararlılık kestirimi elde edilir. Bütün k değerlerinde $\left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H$ için kestirimler birlikte değerlendirilerek

$$\max_{1\le k\le N-1}\left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H\le M\left\{\sum_{s=1}^{N-1}\left\|f_s\right\|_H\tau+\left\|A^{1/2}\varphi\right\|_H+\left\|\psi\right\|_H+\tau\left\|f_{2,2}\right\|_H\right\} \tag{2.96}$$

elde edilir. Böylece ,kestirimler (2.91) ve (2.96) dan yararlanarak, (2.75) kararlılık kestirimi elde edilir. İspat için gerekli diğer bir kararlılık kestirimini elde etmek için, formül (2.94) e Abel formülü uygulanarak

$$\frac{u_{k+1}-u_{k-1}}{2\tau}=\frac{1}{2\tau}\left[J_3R^{k-1}+\widehat{J}_3\tilde{R}^{k-1}\right]\varphi+\frac{1}{2\tau}\left[\widehat{J}_3\tilde{R}^{k-1}-J_3R^{k-1}\right]iA^{-1/2}\psi$$

$$+\frac{1}{2\tau}\left[\tilde{R}^k+R^k\right]\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2 f_{2,2}+\frac{1}{2\tau}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2 f_k$$

$$+\frac{1}{2\tau}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\left[\tilde{R}+R\right]\tau^2 f_{k-1}+\frac{1}{2}\tau\sum_{s=2}^{k-2}\left[\widehat{J}_2\tilde{R}^{k-s}+J_2R^{k-s}\right]\left(f_s-f_{s-1}\right)$$

$$-\frac{1}{2}\tau\left[\widehat{J}_2\widehat{R}+J_2R\right]f_{k-2}+\frac{1}{2}\tau\left[\widehat{J}_2\tilde{R}^{k-1}+J_2R^{k-1}\right]f_1\Big\},\ 2\le k\le N \qquad (2.97)$$

elde edilir. Buradan, formül (2.97) ye $A^{1/2}$ operatörü ve üçgen eşitsizliği uygulanarak ve (2.73) kestirimlerinden yararlanılarak

$$\left\|A^{1/2}\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H$$

$$\le\frac{1}{2\tau}\left(\|A^{-1/2}J_3\|_{H\to H}\|R^{k-1}\|_{H\to H}+\left\|A^{-1/2}\widetilde{J}_3\right\|_{H\to H}\left\|\widetilde{R}^{k-1}\right\|_{H\to H}\right)\|A\varphi\|_H$$

$$+\frac{1}{2}|i|\left(\left\|A^{-1/2}\widetilde{J}_3\right\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H}+\|A^{-1/2}J_3\|_{H\to H}\|R^{k-1}\|_{H\to H}\right)\|A^{1/2}\psi\|_H$$

$$+\frac{1}{2}\left(\|\tilde{R}^k\|_{H\to H}+\|R^k\|_{H\to H}\right)\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\|A^{1/2}f_{2,2}\|_H\tau$$

$$+\frac{1}{2}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\|A^{1/2}f_k\|_H\tau$$

$$+\frac{1}{2}\left(\|\tilde{R}\|_{H\to H}+\|R\|_{H\to H}\right)\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}$$

$$\times\|A^{1/2}f_{k-1}\|_H\tau+\frac{1}{2}\sum_{s=2}^{k-2}\left(\left\|\tau A^{1/2}\widetilde{J}_2\right\|_{H\to H}\|\tilde{R}^{k-s}\|_{H\to H}\right.$$

$$+\|\tau A^{1/2}J_2\|_{H\to H}\|R^{k-s}\|_{H\to H}\big)\|f_s-f_{s-1}\|_H$$

$$+\frac{1}{2}\Big[\,\|\tau A^{1/2}\widetilde{J}_2\|_{H\to H}\|\widetilde{R}\|_{H\to H}+\|\tau A^{1/2}J_2\|_{H\to H}\|R\|_{H\to H}\Big]\|f_{k-2}\|_H$$

$$+\frac{1}{2}\Big[\,\|\tau A^{1/2}\widetilde{J}_2\|_{H\to H}\left\|\widetilde{R}^{k-1}\right\|_{H\to H}+\|\tau A^{1/2}J_2\|_{H\to H}\|R^{k-1}\|_{H\to H}\Big]\|f_1\|_H$$

$$\leq M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\varphi\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{2,2}\right\|_H\right\} \tag{2.97a}$$

kararlılık kestirimi elde edilir. Bütün k değerlerinde $\left\|A^{1/2}\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H$ için kestirimler birlikte değerlendirilerek

$$\max_{1\leq k\leq N-1}\left\|A^{1/2}\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H\leq M\Bigg\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A^{1/2}\psi\|_H$$

$$+\|A\varphi\|_H+\tau\left\|A^{1/2}f_{2,2}\right\|_H\Bigg\} \tag{2.98}$$

kararlılık kestirimi bulundu. Kestirimler (2.93) ve (2.98) yardımı ile (2.76) kararlılık kestirimi elde edilir. Böylece, Teorem 2.5 ispatı tamamlanmış olur. Şimdi Teorem 2.5 in bazı sonuç teoremleri verilecek. Önce (2.64) problemini ele alalım. Problem (2.64) ün ayrışımı iki adımda ele alındı. İlk adım, 2.2 alt başlığındaki adımlarla tamamen aynıdır. İkinci adımda problem (2.66) nın fark şeması

$$\begin{cases}\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+\frac{5}{6}A_h^x u_k^h(x)+\frac{1}{12}A_h^x\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)\\ -\frac{1}{72}\tau^2\left(A_h^x\right)^2u_{k+1}^h(x)+\frac{1}{144}\tau^2\left(A_h^x\right)^2\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)=f_k^h(x),\\ f_k^h(x)=\frac{5}{6}f^h(t_k,x)+\frac{1}{12}\left(f^h(t_{k+1},x)+f^h(t_{k-1},x)\right)\\ +\frac{1}{72}\tau^2\left(-Af^h(t_k,x)+f_{tt}^h(t_k,x)\right)\\ -\frac{1}{144}\tau^2\left(-A\left(f^h(t_{k+1},x)+f^h(t_{k-1},x)\right)\right.\\ \left.+f_{tt}^h(t_{k+1},x)+f_{tt}^h(t_{k-1},x)\right),x\in[0,1]_h,\\ t_k=k\tau,N\tau=1,1\le k\le N-1,\\ u_0^h(x)=\varphi^h(x),x\in[0,1]_h,\\ \left(I+\frac{\tau^2}{12}\left(A_h^x\right)+\frac{\tau^4}{144}\left(A_h^x\right)^2\right)\tau^{-1}\left(u_1^h(x)-u_0^h(x)\right)\\ =\left(-\left(\frac{\tau}{2}\left(A_h^x\right)\right)\varphi^h(x)+\left(I-\frac{\tau^2}{12}\left(A_h^x\right)\right)\psi^h(x)\right.\\ \left.+\tau f_{2,2}(x)\right),x\in[0,1]_h,\\ f_{2,2}(x)=\frac{1}{2}f^h(0,x)+\frac{\tau}{6}f_t^h(0,x)+\frac{\tau^2}{24}f_{tt}^h(0,x)\end{cases}\tag{2.99}$$

yazılır.

Teorem 2.6 τ ve h yeterince küçük sayılar olsunlar. Bu takdirde, fark şeması (2.99) un çözümü aşağıdaki kararlılık kestirimlerini sağlanır;

$$\max_{0\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{L_{2h}}+\max_{0\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^1}+\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-u_{k-1}^h}{2\tau}\right\|_{L_{2h}}$$

$$\le M_1\left[\max_{1\le k\le N-1}\|f_k^h\|_{L_{2h}}+\|\psi^h\|_{L_{2h}}+\|\varphi^h\|_{W_{2h}^1}+\tau\|f_{2,2}\|_{L_{2h}}\right],$$

$$\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-u_{k-1}^h}{2\tau}\right\|_{W_{2h}^1}+\max_{0\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^2}$$

$$\le M_1\left[\left\|f_1^h\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\tau^{-1}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}+\left\|\psi^h\right\|_{W_{2h}^1}+\left\|\varphi^h\right\|_{W_{2h}^2}+\tau\left\|f_{2,2}\right\|_{W_{2h}^1}\right].$$

Burada M_1 sabiti, $\tau,\ h,\ \varphi^h(x),\ \psi^h(x),\ f_{2,2}$ ve $f_k^h\ \left(1\le k<N\right)$ den bağımsızdır.

Teorem 2.6 nın ispatı soyut (abstract) Teorem 2.5 in ispatına ve (2.65) te verilen A_h^x fark operatörünün simetri özeliğine bağlıdır. Şimdi, problem (2.68) i ele alalım. Problem (2.68) in ayrışımı iki adımda yapılabilir. İlk adım, 2.2 alt başlığındaki uygulamanın tamamen aynıdır. İkinci adımda problem (2.70) in fark şeması

$$\begin{cases} \frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+\frac{5}{6}A_h^x u_k^h(x)+\frac{1}{12}A_h^x\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right) \\ -\frac{1}{72}\tau^2\left(A_h^x\right)^2 u_{k+1}^h(x)+\frac{1}{144}\tau^2\left(A_h^x\right)^2\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)=f_k^h(x), \\ f_k^h(x)=\frac{5}{6}f^h(t_k,x)+\frac{1}{12}\left(f^h(t_{k+1},x)+f^h(t_{k-1},x)\right) \\ +\frac{1}{72}\tau^2\left(-Af^h(t_k,x)+f_{tt}^h(t_k,x)\right) \\ -\frac{1}{144}\tau^2\left(-A\left(f^h(t_{k+1},x)+f^h(t_{k-1},x)\right)\right. \\ \left.+f_{tt}^h(t_{k+1},x)+f_{tt}^h(t_{k-1},x)\right),x\in\Omega_h, \\ t_k=k\tau,N\tau=1,1\le k\le N-1, \\ u_0^h(x)=\varphi^h(x),x\in\widehat{\Omega}_h, \\ \left(I+\frac{\tau^2}{12}\left(A_h^x\right)+\frac{\tau^4}{144}\left(A_h^x\right)^2\right)\tau^{-1}\left(u_1^h(x)-u_0^h(x)\right) \\ =\left(-\left(\frac{\tau}{2}\left(A_h^x\right)\right)\varphi^h(x)+\left(I-\frac{\tau^2}{12}\left(A_h^x\right)\right)\psi^h(x)\right. \\ \left.+\tau f_{2,2}(x)\right),x\in\widehat{\Omega}_h, \\ f_{2,2}(x)=\frac{1}{2}f^h(0,x)+\frac{\tau}{6}f_t^h(0,x)+\frac{\tau^2}{24}f_{tt}^h(0,x) \end{cases} \tag{2.100}$$

yazılır.

Teorem 2.7 τ ve $|h|$ yeterince küçük sayılar olsun. Bu takdirde fark şeması (2.100) ün çözümü için aşağıdaki kararlılık kestirimleri sağlanır;

$$\max_{0\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{L_{2h}}+\max_{0\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^1}+\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-u_{k-1}^h}{2\tau}\right\|_{L_{2h}}$$

$$\le M_1\left[\max_{1\le k\le N-1}\|f_k^h\|_{L_{2h}}+\|\psi^h\|_{L_{2h}}+\|\varphi^h\|_{W_{2h}^1}+\tau\|f_{2,2}\|_{L_{2h}}\right],$$

$$\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-u_{k-1}^h}{2\tau}\right\|_{W_{2h}^1}+\max_{0\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^2}\le M_1\left[\left\|f_1^h\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\tau^{-1}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}\right.$$

$$\left.+\left\|\psi^h\right\|_{W_{2h}^1}+\left\|\varphi^h\right\|_{W_{2h}^2}+\tau\left\|f_{2,2}\right\|_{W_{2h}^1}\right].$$

Burada M_1 sabiti $\tau,\ h,\ \varphi^h(x),\ \psi^h(x)$, $f_{2,2}^h$ ve $f_k^h(1\le k<N)$ ifadelerinden bağımsızdır.

Teorem 2.7 nin ispatı Teorem 2.5 in ispatına, (2.69) da tanımlanan A_h^x fark operatörünün simetri özeliğine ve Teorem 2.4 e bağlıdır.

3. LOKAL OLMAYAN HİPERBOLİK TİP SINIR DEĞER PROBLEMİ İÇİN KARARLI FARK ŞEMALARI

3.1 Lokal Olmayan Hiperbolik Tip Sınır Değer Problemi

Bu bölümde, Hilbert uzayında lokal olmayan, kendine eşlenik, pozitif tanımlı A operatörü yardımı ile oluşturulan hiperbolik sınır değer problemi

$$\begin{cases} u''(t) + Au(t) = f(t), 0 < t < T, \\ u(0) = \alpha u(1) + \varphi,\ u'(0) = \beta u'(1) + \psi \end{cases} \tag{3.1}$$

ele alındı. Problem (3.1) in çözümü için kararlılık kestirimleri elde edildi. Ayrıca problemin yaklaşık çözümünü elde etmek için A operatörünün tam kuvvetleri kullanılarak üçüncü ve dördüncü mertebeden doğruluk fark şemaları ve bu fark şemalarının çözümleri için kararlılık kestirimleri elde edildi.

3.2 Lokal Olmayan Hiperbolik Tip Sınır Değer Problemi İçin Üçüncü Mertebeden Kararlı Doğruluk Fark Şeması

Bu kısımda, lokal olmayan sınır değer problemi (3.1) in nümerik çözümü için, üçüncü mertebeden doğruluk fark şeması elde edildi. Fark şeması (2.6) ve (2.19) formülü kullanılarak, problem (3.1) in çözümü için üçüncü mertebeden iki adımlı fark şeması

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{2}{3}Au_k + \frac{1}{6}A\left(u_{k+1} + u_{k-1}\right) \\ + \frac{1}{12}\tau^2 A^2 u_{k+1} = f_k,\ f_k = \frac{2}{3}f(t_k) + \frac{1}{6}\left(f\left(t_{k+1}\right) + f\left(t_{k-1}\right)\right) \\ - \frac{1}{12}\tau^2\left(-Af\left(t_{k+1}\right) + f''(t_{k+1})\right), t_k = k\tau, 1 \le k \le N-1, N\tau = 1, \\ u_0 = \alpha u_N + \varphi, \left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)\tau^{-1}\left(u_1 - u_0\right) + \frac{\tau}{2}Au_0 - \tau f_{1,1} \\ = \beta\left(I - \frac{\tau^2 A}{12}\right)\left(\dfrac{7u_N - 8u_{N-1} + u_{N-2}}{6\tau} + \frac{\tau}{3}\left(f_N - Au_N\right)\right) + \left(I - \frac{\tau^2 A}{12}\right)\psi, \end{cases} \tag{3.2}$$

elde edildi. Burada

$$f_{1,1} = \left(I + \frac{\tau^2}{12} A + \frac{\tau^4}{144} A^2 \right) \tau^{-1} f_{1,1}^3 = \left\{ f(0) + \left(-f(0) + \tau f'(0) \right) \frac{1}{2} - 2 f'(0) \frac{\tau}{6} \right\}$$

dır. Fark şeması (3.2) nin çözümünün kararlılığı

$$|\alpha| + 2|\beta| + 2|\alpha||\beta| < 1 \tag{3.3}$$

koşulu altında incelendi. Bu kısımda, kolaylık olması açısından

$$B_n^\tau = \beta \frac{1}{2} R_2 \left(\widehat{R}_7 \widehat{R}_5 - \frac{\tau A}{3} \widehat{R}^2 \right) \tilde{R}^{N-2} + \beta \frac{1}{2} R_2 \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) R^{N-2} - \alpha \frac{1}{2} \left[\widehat{R}_1 R^N - R_1 \tilde{R}^N \right]$$

$$+ \alpha\beta \frac{1}{4} \tilde{R}_1 R_2 \left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) R^N \tilde{R}^{N-2} + \alpha\beta \frac{1}{4} R_1 R_2 \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) \tilde{R}^N R^{N-2}$$

$$- \alpha\beta \frac{1}{4} \widehat{R}_1 R_2 \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) \tilde{R}^N R^{N-2} - \alpha\beta \frac{1}{4} R_1 R_2 \left(\widehat{R}_7 \widehat{R}_5 - \frac{\tau A}{3} \widehat{R}^2 \right) R^N \tilde{R}^{N-2}$$

eşitliği kullanılacak. Bu kısımda, (2.25), (2.26), (2.27), (2.28), (2.29), (2.30) operatörlerinden, Lemma 2.1 de sunulan kestirimlerden ve

$$R_7 = \frac{(7R - I)}{6\tau} = \left(I - \frac{5}{12} \tau^2 A + \frac{1}{72} \tau^4 A^2 + \frac{7}{6} i \tau A^{1/2} \sqrt{I + \frac{1}{72} \tau^4 A^2} \right)$$

$$\times \tau^{-1} \left(I + \frac{1}{6} \tau^2 A + \frac{1}{12} \tau^4 A^2 \right)^{-1}, \tag{3.4}$$

ve eşleniği $\tilde{R}_7$,

$$\widetilde{R}_7 = \frac{\left(7\widetilde{R} - I\right)}{6\tau} = \left(I - \frac{5}{12}\tau^2 A + \frac{1}{72}\tau^4 A^2 - \frac{7}{6} i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\right)$$

$$\times \tau^{-1}\left(I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2\right)^{-1}, \tag{3.5}$$

$$R_8 = \left(\frac{7I - 2\tau^2 A}{6\tau}\right)\left(I + \frac{\tau^2 A}{3} + \frac{\tau^4 A^2}{9} + \frac{\tau^6 A^3}{72}\right)$$

$$\times \tau^{-1}\left(I + \frac{\tau^2 A}{6}\right)^{-1}\left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2\right)^{-2}, \tag{3.6}$$

$$R_9 = \left(I - \frac{5}{3}\tau^2 A + \frac{\tau^4 A^2}{9}\right)\left(I + \frac{\tau^2 A}{3} + \frac{\tau^4 A^2}{9} + \frac{\tau^6 A^3}{72}\right)$$

$$\times \tau^{-1}\left(I + \frac{\tau^2 A}{6}\right)^{-1}\left(I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2\right)^{-3}, \tag{3.7}$$

$$R_{10} = I + \left(\frac{5}{144}\tau^4 A^2 - \frac{9}{288}\tau^6 A^3 + \frac{9}{1728}\tau^8 A^4\right)$$

$$\times \left(i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)\right)^{-1}, \tag{3.8}$$

ve eşleniği $\widehat{R}_{10}$ operatörlerinden yararlanıldı. Önce, teoremin ispatında yararlanılacak iki lemmayı sırasıyla ele alalım.

Lemma 3.1 Aşağıdaki kararlılık kestirimleri sağlanır.

$$\begin{cases} \left\| \left(I + i\tau A^{1/2}\right) R \right\|_{H \to H} \le 2, \left\| \left(I + i\tau A^{1/2}\right) \widehat{R} \right\|_{H \to H} \le 2, \\ \left\| \tau R_7 \right\|_{H \to H} \le 1, \left\| \tau \widehat{R}_7 \right\|_{H \to H} \le 1, \\ \left\| \frac{1}{3}\tau A^{1/2} R^2 \right\|_{H \to H} \le 1, \left\| \frac{1}{3}\tau A^{1/2} \widehat{R}^2 \right\|_{H \to H} \le 1, \\ \left\| \tau R_8 \right\|_{H \to H} \le \frac{7}{6}, \left\| \tau R_9 \right\|_{H \to H} \le 1, \\ \left\| R_{10}\left(I + i\tau A^{1/2}\right)^{-1} \right\|_{H \to H} \le 2, \left\| \widehat{R}_{10}\left(I + i\tau A^{1/2}\right)^{-1} \right\|_{H \to H} \le 2. \end{cases} \tag{3.9}$$

İspat: Lemmanın ispatında (2.31) de verilen kestirimlerden yararlanıldı ve Lemma 2.1 deki adımlar takip edildi. Kendine eşlenik ve pozitif tanımlı operatörün spektral özeliği yardımı ile

$$\left\| \tau\left(I - \frac{5}{12}\tau^2 A + \frac{1}{72}\tau^4 A^2 \pm \frac{7}{6} i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\right)\tau^{-1}\left(I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2\right)^{-1} \right\|_{H\to H}$$

$$\leq \sup_{\delta\leq\lambda<\infty}\left|\frac{1 - \frac{5}{12}\tau^2\lambda + \frac{1}{72}\tau^4\lambda^2 \pm \frac{7}{6} i\tau\lambda^{1/2}\sqrt{1 + \frac{1}{72}\tau^4\lambda^2}}{1 + \frac{1}{6}\tau^2\lambda + \frac{1}{12}\tau^4\lambda^2}\right|$$

eşitsizliği yazıldı. Burada da

$$\frac{\left(1 - \frac{5}{12}\tau^2\lambda + \frac{1}{72}\tau^4\lambda^2\right)^2 + \left(\frac{7}{6}\tau\lambda^{1/2}\sqrt{1 + \frac{1}{72}\tau^4\lambda^2}\right)^2}{\left(1 + \frac{1}{6}\tau^2\lambda + \frac{1}{12}\tau^4\lambda^2\right)^2} \leq 1$$

olduğundan, $\left\|\tau R_7\right\|_{H\to H} \leq 1$, $\left\|\tau \widehat{R}_7\right\|_{H\to H} \leq 1$ dir. Aynı şekilde

$$\left\| \frac{\tau A^{1/2}}{3}\left(I - \frac{1}{3}\tau^2 A \pm i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\right)^2\left(I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2\right)^{-2} \right\|_{H\to H}$$

$$\leq \sup_{\delta\leq\lambda<\infty}\left|\frac{\frac{\tau\lambda^{1/2}}{3}\left(1 - \frac{1}{3}\tau^2\lambda \pm i\tau\lambda^{1/2}\sqrt{1 + \frac{1}{72}\tau^4\lambda^2}\right)^2}{\left(1 + \frac{1}{6}\tau^2\lambda + \frac{1}{12}\tau^4\lambda^2\right)^2}\right|$$

yazılabilir. Buradan da

$$\frac{\frac{\tau^2\lambda}{9}\left(\left(1 - \frac{5}{3}\tau^2\lambda + \frac{1}{9}\tau^4\lambda^2 - \frac{1}{72}\tau^6\lambda^3\right)^2 + \frac{4}{9}\tau^6\lambda^3\left(1 + \frac{1}{72}\tau^4\lambda^2\right)\right)}{\left(1 + \frac{1}{3}\tau^2\lambda + \frac{7}{36}\tau^4\lambda^2 + \frac{1}{36}\tau^6\lambda^3 + \frac{1}{144}\tau^8\lambda^4\right)^2} \leq 1$$

olduğundan, $\left\| \frac{1}{3}\tau A^{1/2} R^2 \right\|_{H\to H} \le 1,\ \left\| \frac{1}{3}\tau A^{1/2} \widehat{R}^2 \right\|_{H\to H} \le 1$ dir. Benzer şekilde

$$\left\| \tau \left(\frac{7I - 2\tau^2 A}{6\tau} \right) \left(I + \frac{\tau^2 A}{3} + \frac{\tau^4 A^2}{9} + \frac{\tau^6 A^3}{72} \right) \right.$$

$$\left. \times \tau^{-1} \left(I + \frac{\tau^2 A}{6} \right)^{-1} \left(I + \frac{\tau^2}{6} A + \frac{\tau^4}{12} A^2 \right)^{-2} \right\|_{H\to H} \le \frac{7}{6}$$

yazılabilir. Buradan $\left\| \tau R_8 \right\|_{H\to H} \le 7/6$ dır. Aynı adımlarla

$$\left\| \tau \left(I - \frac{5}{3}\tau^2 A + \frac{\tau^4 A^2}{9} \right) \left(I + \frac{\tau^2 A}{3} + \frac{\tau^4 A^2}{9} + \frac{\tau^6 A^3}{72} \right) \right.$$

$$\left. \times \tau^{-1} \left(I + \frac{\tau^2 A}{6} \right)^{-1} \left(I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2 \right)^{-3} \right\|_{H\to H} \le 1$$

yazılabilir. Buradan $\left\| \tau R_9 \right\|_{H\to H} \le 1$ dir. Benzer şekilde

$$\left\| \left(\frac{5}{144}\tau^4 A^2 - \frac{9}{288}\tau^6 A^3 + \frac{9}{1\,728}\tau^8 A^4 \pm i\tau A^{1/2} \left(I + \frac{\tau^2}{12} A + \frac{\tau^4}{144} A^2 \right) \sqrt{I + \frac{1}{72}\tau^4 A^2} \right) \right.$$

$$\left. \times \left((I + i\tau A^{1/2}) i\tau A^{1/2} \sqrt{I + \frac{1}{72}\tau^4 A^2} \left(I + \frac{\tau^2}{12} A + \frac{\tau^4}{144} A^2 \right) \right)^{-1} \right\|_{H\to H}$$

$$\le \sup_{\delta \le \lambda < \infty} \left| \frac{\frac{5}{144}\tau^4\lambda^2 - \frac{9}{288}\tau^6\lambda^3 + \frac{9}{1728}\tau^8\lambda^4 + i\tau\lambda^{1/2}\left(1 + \frac{\tau^2}{12}\lambda + \frac{\tau^4}{144}\lambda^2\right)\sqrt{1 + \frac{1}{72}\tau^4\lambda^2}}{\left(1 + i\tau\lambda^{1/2}\right) i\tau\lambda^{1/2} \sqrt{1 + \frac{1}{72}\tau^4\lambda^2} \left(1 + \frac{\tau^2}{12}\lambda + \frac{\tau^4}{144}\lambda^2\right)} \right|$$

yazılabilir. Buradan da

$$\frac{\left(\frac{5\tau^4\lambda^2}{144}-\frac{9\tau^6\lambda^3}{288}+\frac{9\tau^8\lambda^4}{1728}\right)^2+\left(\tau\lambda^{1/2}\left(1+\frac{\tau^2\lambda}{12}+\frac{\tau^4\lambda^2}{144}\right)\sqrt{1+\frac{\tau^4\lambda^2}{72}}\right)^2}{\left(\tau\lambda^{1/2}\sqrt{1+\frac{\tau^4\lambda^2}{72}}\left(1+\frac{\tau^2\lambda}{12}+\frac{\tau^4\lambda^2}{144}\right)\right)^2\left(1+\tau^2\lambda\right)}\le 2$$

olduğundan, $\left\| R_{10}\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\le 2$, $\left\| \widehat{R}_{10}\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\le 2$ dir ve aynı yolla $\left\|\left(I+i\tau A^{1/2}\right)R\right\|_{H\to H}\le 2$ ve $\left\|\left(I+i\tau A^{1/2}\right)\widehat{R}\right\|_{H\to H}\le 2$ dir. Bu Lemma 3.1 i ispat eder.

Lemma 3.2 Koşul (3.3) sağlansın, bu takdirde $I-B_n^\tau$ operatörünün sınırlı bir tersi $T_\tau=\left(I-B_n^\tau\right)^{-1}$ mevcuttur. A operatörünün simetri özeliği ve pozitif tanımlılığı yardımıyla

$$\left\|T_\tau\right\|_{H\mapsto H}\le\frac{1}{1-|\alpha|-2|\beta|-2|\alpha||\beta|} \tag{3.10}$$

eşitsizliği yazılabilir.

İspat: B_n^τ, R, $\tilde{R}$, operatörlerinden ve (2.31), (2.39), (2.40) kestirimlerinden yararlanılır, üçgen eşitsizliğini uygulanırsa

$$B_n^\tau\le|\beta|\frac{1}{2}\|A^{1/2}R_2\|_{H\to H}\|\tilde{R}^{N-2}\|_{H\to H}$$

$$\times\left(\|\tau\widetilde{R}_7\|_{H\to H}\|\tau^{-1}A^{-1/2}\widetilde{R}_5\|_{H\to H}+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\right)+|\beta|\frac{1}{2}\|A^{1/2}R_2\|_{H\to H}$$

$$\times\|R^{N-2}\|_{H\to H}\left(\|\tau R_7\|_{H\to H}\|\tau^{-1}A^{-1/2}R_5\|_{H\to H}+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\right)$$

$$+\alpha\frac{1}{2}\left[\|\widetilde{R}_1\|_{H\to H}\|R^N\|_{H\to H}+\|R_1\|_{H\to H}\|\tilde{R}^N\|_{H\to H}\right]$$

$$+|\alpha||\beta|\frac{1}{4}\left\|\widetilde{R}_1\right\|_{H\to H}\|A^{1/2}R_2\|_{H\to H}\|R^N\|_{H\to H}\|\tilde{R}^{N-2}\|_{H\to H}$$

$$\times\left(\left\|\tau\widetilde{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\right)$$

$$+|\alpha||\beta|\frac{1}{4}\|R_1\|_{H\to H}\|A^{1/2}R_2\|_{H\to H}\|\tilde{R}^N\|_{H\to H}\|R^{N-2}\|_{H\to H}$$

$$\times\left(\|\tau R_7\|_{H\to H}\|\tau^{-1}A^{-1/2}R_5\|_{H\to H}+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\right)$$

$$+|\alpha||\beta|\frac{1}{4}\left\|\widetilde{R}_1\right\|_{H\to H}\|A^{1/2}R_2\|_{H\to H}\|\tilde{R}^N\|_{H\to H}\|R^{N-2}\|_{H\to H}$$

$$\times\left(\|\tau R_7\|_{H\to H}\|\tau^{-1}A^{-1/2}R_5\|_{H\to H}+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\right)$$

$$+|\alpha||\beta|\frac{1}{4}\|R_1\|_{H\to H}\|A^{1/2}R_2\|_{H\to H}\|R^N\|_{H\to H}\|\tilde{R}^{N-2}\|_{H\to H}$$

$$\times\left(\left\|\tau\widetilde{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\right)\leq q$$

eşitsizliği elde edilir. Burada $q=|\alpha|+2|\beta|+2|\alpha||\beta|$ dır. $q<1$ halinde, $I-B_n^\tau$ operatörünün sınırlı bir tersi

$$\left\|\left(I-B_n^\tau\right)^{-1}\right\|_{H\to H}\leq\frac{1}{1-q}=\frac{1}{1-|\alpha|-2|\beta|-2|\alpha||\beta|}$$

mevcuttur. Bu Lemma 3.2 yi ispat eder.

Şimdi problem (3.2) için çözüm formülü elde edelim. 2.1 başlığındaki adımlar uygulanarak, fark şeması

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{2}{3}Au_k+\frac{1}{6}A\left(u_{k+1}+u_{k-1}\right) \\ +\frac{1}{12}\tau^2A^2u_{k+1}=f_k,\ f_k=\frac{2}{3}f(t_k)+\frac{1}{6}\left(f\left(t_{k+1}\right)+f\left(t_{k-1}\right)\right) \\ -\frac{1}{12}\tau^2\left(-Af\left(t_{k+1}\right)+f''(t_{k+1})\right), t_k=k\tau, 1\le k\le N-1, N\tau=1, \\ u_0=\mu, \left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\tau^{-1}\left(u_1-u_0\right)+\frac{\tau}{2}Au_0-\tau f_{1,1} \\ =\left(I-\frac{\tau^2A}{12}\right)\omega \end{cases} \tag{3.11}$$

in çözüm formülü

$$u_0=\mu, u_1=\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)^{-1}$$

$$\times\left(\left(I-\frac{5}{12}\tau^2A+\frac{\tau^4}{144}A^2\right)\mu+\tau\left(I-\frac{\tau^2}{12}A\right)\omega+\tau^2f_{1,1}\right),$$

$$u_k=\frac{1}{2}\left[\widehat{R}_{10}R^k-R_{10}\tilde{R}^k\right]\mu+\frac{1}{2}\left[\tilde{R}^k-R^k\right]R_2\omega$$

$$+\frac{1}{2}\left[\tilde{R}^k-R^k\right]R_3\tau^2f_{1,1}+\frac{1}{2}R_4\sum_{s=1}^{k-1}\left[\tilde{R}^{k-s}-R^{k-s}\right]f_s\tau^2, \tag{3.12}$$

elde edilir. Formül (3.12) ve lokal olmayan sınır şartları

$$\begin{cases} u_0=\alpha u_N+\varphi, \\ \omega=\beta\left(\frac{7u_N-8u_{N-1}+u_{N-2}}{6\tau}+\frac{\tau}{3}(f_N-Au_N)\right)+\psi, \end{cases}$$

kullanılarak

$$\mu=\alpha\left\{\frac{1}{2}\left[\widetilde{R}_{10}R^N-R_{10}\tilde{R}^N\right]\mu+\frac{1}{2}[\tilde{R}^N-R^N]R_2\omega\right.$$

$$+\frac{1}{2}\left[\tilde{R}^{N}-R^{N}\right]R_{3}\tau^{2}f_{1,1}+\frac{1}{2}R_{4}\sum_{s=1}^{N-1}\left[\tilde{R}^{N-s}-R^{N-s}\right]f_{s}\tau^{2}\Bigg\}+\varphi, \tag{3.13}$$

ve

$$\omega=\beta\Bigg\{\frac{1}{2}\Bigg[\left(R_{7}R_{5}-\frac{\tau A}{3}R^{2}\right)\widehat{R}_{10}R^{N-2}-\left(\widehat{R}_{7}\widehat{R}_{5}-\frac{\tau A}{3}\widehat{R}^{2}\right)R_{10}\tilde{R}^{N-2}\Bigg]\mu$$

$$+\frac{1}{2}\left[\left(\widetilde{R}_{7}\widetilde{R}_{5}-\frac{\tau A}{3}\widetilde{R}^{2}\right)\tilde{R}^{N-2}-\left(R_{7}R_{5}-\frac{\tau A}{3}R^{2}\right)R^{N-2}\right]R_{2}\omega$$

$$+\frac{1}{2}\left[\left(\widetilde{R}_{7}\widetilde{R}_{5}-\frac{\tau A}{3}\widetilde{R}^{2}\right)\tilde{R}^{N-2}-\left(R_{7}R_{5}-\frac{\tau A}{3}R^{2}\right)R^{N-2}\right]R_{3}\tau^{2}f_{1,1}$$

$$+\frac{\tau}{3}f_{N}+\frac{1}{2}R_{8}f_{N-1}\tau^{2}+\frac{1}{2}R_{9}f_{N-2}\tau^{2}$$

$$+\frac{1}{2}R_{4}\tau\sum_{s=1}^{N-3}\Bigg[\left(\widehat{R}_{7}\widehat{R}_{5}-\frac{\tau A}{3}\widehat{R}^{2}\right)\tilde{R}^{N-2-s}-\left(R_{7}R_{5}-\frac{\tau A}{3}R^{2}\right)R^{N-2-s}\Bigg]f_{s}\tau\Bigg\}+\psi \tag{3.14}$$

elde edilir. Formüller (3.13) ve (3.14) yardımı ile

$$\mu=T_{\tau}\Bigg\{\Bigg[\alpha\Bigg(\frac{1}{2}(\tilde{R}^{N}-R^{N})R_{3}\tau^{2}f_{1,1}+\frac{1}{2}R_{4}\tau\sum_{s=1}^{N-1}(\tilde{R}^{N-s}-R^{N-s})f_{s}\tau\Bigg)+\varphi\Bigg]$$

$$\times\left[I-\frac{1}{2}\left(\left(\widetilde{R}_{7}\widetilde{R}_{5}-\frac{\tau A}{3}\widetilde{R}^{2}\right)\tilde{R}^{N-2}-\left(R_{7}R_{5}-\frac{\tau A}{3}R^{2}\right)R^{N-2}\right)R_{2}\right]$$

$$+\alpha\frac{1}{2}\left(\tilde{R}^{N}-R^{N}\right)R_{2}\Bigg[\beta\frac{1}{2}\Bigg\{\Bigg(\left(\widehat{R}_{7}\widehat{R}_{5}-\frac{\tau A}{3}\widehat{R}^{2}\right)\tilde{R}^{N-2}-\left(R_{7}R_{5}-\frac{\tau A}{3}R^{2}\right)R^{N-2}\Bigg)$$

$$\times R_{3}\tau^{2}f_{1,1}+\frac{2\tau}{3}f_{N}+R_{8}f_{N-1}\tau^{2}+R_{9}f_{N-2}\tau^{2}$$

$$+R_4\tau\sum_{s=1}^{N-3}\left[\left(\widehat{R}_7\widehat{R}_5-\frac{\tau A}{3}\widehat{R}^2\right)\tilde{R}^{N-2-s}-\left(R_7R_5-\frac{\tau A}{3}R^2\right)R^{N-2-s}\right]f_s\tau\Big\}+\psi\Big] \tag{3.15}$$

ve

$$\omega = T_\tau\Big\{\Big[I-\alpha\frac{1}{2}\left(\widetilde{R}_{10}R^N-R_{10}\tilde{R}^N\right)\Big]$$

$$\times\Big[\beta\frac{1}{2}\Big\{\Big(\left(\widetilde{R}_7\widetilde{R}_5-\frac{\tau A}{3}\widetilde{R}^2\right)\tilde{R}^{N-2}-\left(R_7R_5-\frac{\tau A}{3}R^2\right)R^{N-2}\Big)$$

$$\times R_3\tau^2 f_{1,1}+\frac{2\tau}{3}f_N+R_8f_{N-1}\tau^2+R_9f_{N-2}\tau^2+R_4\tau$$

$$\times\sum_{s=1}^{N-3}\Big(\left(\widetilde{R}_7\widetilde{R}_5-\frac{\tau A}{3}\widetilde{R}^2\right)\tilde{R}^{N-2-s}-\left(R_7R_5-\frac{\tau A}{3}R^2\right)R^{N-2-s}\Big)f_s\tau\Big\}+\psi\Big]$$

$$+\frac{1}{2}\Big[\left(R_7R_5-\frac{\tau A}{3}R^2\right)\widetilde{R}_{10}R^{N-2}+\left(\widetilde{R}_7\widetilde{R}_5-\frac{\tau A}{3}\widetilde{R}^2\right)R_{10}\tilde{R}^{N-2}\Big]$$

$$\times\left[\alpha\left(\frac{1}{2}\left(\tilde{R}^N-R^N\right)R_3\tau^2 f_{1,1}+\frac{1}{2}R_4\tau\sum_{s=1}^{N-1}\left(\tilde{R}^{N-s}-R^{N-s}\right)f_s\tau\right)+\varphi\right]\Big\} \tag{3.16}$$

bulunur. Böylece, (3.12), μ ve ω sırasıyla (3.15) ve (3.16) da verilmiş olmak üzere, (3.2) probleminin bir çözümüdür.

Ne yazık ki, $\max\limits_{1\le k\le N}\|u_k\|_H$, $\max\limits_{1\le k\le N}\left\|A^{1/2}u_k\right\|_H$, $\max\limits_{1\le k\le N}\|Au_k\|_H$ ifadeleri için kararlılık kestirimleri,

$$\max_{1\le k\le N}\|u_k\|_H\le M\left\{\sum_{s=1}^{k-1}\|A^{-1/2}f_s\|_H\tau+\|\varphi\|_H+\|A^{-1/2}\psi\|_H+\tau\|A^{-1/2}f_{1,1}\|_H\right\},$$

$$\max_{1\le k\le N} \|A^{1/2}u_k\|_H \le M\left\{\sum_{s=1}^{k-1} \|f_s\|_H\tau + \|A^{1/2}\varphi\|_H + \|\psi\|_H + \tau\|f_{1,1}\|_H\right\},$$

$$\max_{1\le k\le N}\|Au_k\|_H \le M\left\{\sum_{s=2}^{N-1}\|f_s - f_{s-1}\|_H] + \|f_1\|_H + \|A\varphi\|_H + \left\|A^{1/2}\psi\right\|_H + \tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}$$

şartları altında elde edilemiyor. Bununla beraber aşağıdaki teorem ispatlanabilir.

Teorem 3.1 $\varphi\in D(A^{3/2})$, $\psi\in D(A^{\frac{1}{2}})$ olsun ve (3.3) koşulu ile Teorem 2.1 deki bütün kararlılık kestirimleri sağlansın. Bu durumda fark şeması (3.2) nin çözümü için aşağıdaki kararlılık kestirimleri sağlanır;

$$\max_{1\le k\le N}\|u_k\|_H \le M\left\{\sum_{s=1}^{k-1}\left\|A^{-1/2}f_s\right\|_H\tau + \left\|\left(I+i\tau A^{1/2}\right)\varphi\right\|_H + \left\|A^{-1/2}\psi\right\|_H + \tau\left\|A^{-1/2}f_{1,1}\right\|_H\right\} \quad (3.17)$$

$$\max_{1\le k\le N}\left\|A^{1/2}u_k\right\|_H \le M\left\{\sum_{s=1}^{k-1}\|f_s\|_H\tau + \left\|A^{1/2}\left(I+i\tau A^{1/2}\right)\varphi\right\|_H + \|\psi\|_H + \tau\left\|f_{1,1}\right\|_H\right\} \quad (3.18)$$

$$\max_{1\le k\le N} \|Au_k\|_H \le M\left\{\sum_{s=2}^{N-1}\|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A(I+i\tau A^{1/2})\varphi\|_H\right.$$

$$\left. + \left\|A^{1/2}\psi\right\|_H + \tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}. \quad (3.19)$$

Burada M sabiti, τ, φ, ψ , $f_{1,1}$ ve f_s $(1\le s\le N-1)$ifadelerinden bağımsızdır.

İspat: (3.15), (3.16) formülleri ve (2.31), (3.9), (3.10) kestirimleri kullanılarak

$$\left\|\left(I+i\tau A^{1/2}\right)\mu\right\|_H \le \|T_\tau\|_{H\to H}$$

$$\times\left\{\left[|\alpha|\left(\frac{1}{2}\left(\left\|\left(I+i\tau A^{1/2}\right)\tilde{R}^N\right\|_{H\to H} + \left\|\left(I+i\tau A^{1/2}\right)R^N\right\|_{H\to H}\right)\left\|\tau A^{1/2}R_3\right\|_{H\to H}\tau\left\|A^{-1/2}f_{1,1}\right\|_H\right.\right.\right.$$

$$+\frac{1}{2}\left\|\tau A^{1/2}R_4\right\|_{H\to H}\sum_{s=1}^{N-1}\left(\left\|\left(I+i\tau A^{1/2}\right)\tilde{R}^{N-s}\right\|_{H\to H}+\left\|\left(I+i\tau A^{1/2}\right)R^{N-s}\right\|_{H\to H}\right)$$

$$\times\|A^{-1/2}f_s\|_{H\to H}\tau\Big) + \|(I+i\tau A^{1/2})\varphi\|_H\Big]$$

$$\times\Big[1+\frac{1}{2}\Big(\Big(\|\tau\widetilde{R}_7\|_{H\to H}\|\tau^{-1}A^{-1/2}\widetilde{R}_5\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\Big\|_{H\to H}\Big)\|\tilde{R}^{N-2}\|_{H\to H}$$

$$+\Big(\|\tau R_7\|_{H\to H}\|\tau^{-1}A^{-1/2}R_5\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{1/2}R^2\Big\|_{H\to H}\Big)\|R^{N-2}\|_{H\to H}\Big)\|A^{1/2}R_2\|_{H\to H}\Big]$$

$$+|\alpha|\frac{1}{2}\Big(\|(I+i\tau A^{1/2})\tilde{R}^N\|_{H\to H}+\|(I+i\tau A^{1/2})R^N\|_{H\to H}\Big)\|A^{1/2}R_2\|_{H\to H}$$

$$\times\Big[|\beta|\frac{1}{2}\Big\{\Big(\Big(\Big(\|\tau\widetilde{R}_7\|_{H\to H}\|\tau^{-1}A^{-1/2}\widetilde{R}_5\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\Big\|_{H\to H}\Big)\|\tilde{R}^{N-2}\|_{H\to H}$$

$$+\Big(\|\tau R_7\|_{H\to H}\|\tau^{-1}A^{-1/2}R_5\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{1/2}R^2\Big\|_{H\to H}\Big)\|R^{N-2}\|_{H\to H}\Big)$$

$$\times\|\tau A^{1/2}R_3\|_{H\to H}\tau\|A^{-1/2}f_{1,1}\|_H+\frac{2\tau}{3}\|A^{-1/2}f_N\|_H$$

$$+\|\tau R_8\|_{H\to H}\|A^{-1/2}f_{N-1}\|_H\tau+\|\tau R_9\|_{H\to H}\|A^{-1/2}f_{N-2}\|_H\tau+\|\tau A^{1/2}R_4\|_{H\to H}$$

$$\times\sum_{s=1}^{N-3}\Big(\Big(\|\tau\widetilde{R}_7\|_{H\to H}\|\tau^{-1}A^{-1/2}\widetilde{R}_5\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\Big\|_{H\to H}\Big)\|\tilde{R}^{N-2-s}\|_{H\to H}$$

$$+\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\right)\left\|R^{N-2-s}\right\|_{H\to H}\Bigg)\left\|A^{-1/2}f_s\right\|_H\tau\Bigg\}+\left\|A^{-1/2}\psi\right\|_H\Bigg]$$

$$\leq M\left\{\sum_{s=1}^{k-1}\left\|A^{-1/2}f_s\right\|_H\tau+\left\|\left(I+i\tau A^{1/2}\right)\varphi\right\|_H+\left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{1,1}\right\|_H\right\}\tag{3.20}$$

ve

$$\left\|A^{-\frac{1}{2}}\omega\right\|_H \le \left\|T_\tau\right\|_{H\to H}\left\{\left[1+|\alpha|\frac{1}{2}\left(\left\|\left(I+i\tau A^{1/2}\right)^{-1}\widehat{R}_{10}\right\|_{H\to H}\left\|\left(I+i\tau A^{1/2}\right)R^N\right\|_{H\to H}\right.\right.\right.$$

$$\left.\left.+\left\|(I+i\tau A^{1/2})^{-1}R_{10}\right\|_{H\to H}\left\|(I+i\tau A^{1/2})\widetilde{R}^N\right\|_{H\to H}\right)\right]$$

$$\times\left[|\beta|\frac{1}{2}\left\{\left(\left(\left(\|\tau\widetilde{R}_7\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\widetilde{R}_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}\widetilde{R}^2\right\|_{H\to H}\right)\|\tilde{R}^{N-2}\|_{H\to H}\right.\right.\right.\right.$$

$$\left.+\left(\|\tau R_7\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}R^2\right\|_{H\to H}\right)\|R^{N-2}\|_{H\to H}\right)$$

$$\times\left\|\tau A^{\frac{1}{2}}R_3\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{1,1}\right\|_H\tau+\frac{2}{3}\left\|A^{-\frac{1}{2}}f_N\right\|_H\tau+\|\tau R_8\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{N-1}\right\|_H\tau$$

$$+\|\tau R_9\|_{H\to H}\left\|A^{-\frac{1}{2}}f_{N-2}\right\|_H\tau+\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}$$

$$\times\sum_{s=1}^{N-3}\left(\left(\|\tau\widetilde{R}_7\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\widetilde{R}_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}\widetilde{R}^2\right\|_{H\to H}\right)\|\tilde{R}^{N-2-s}\|_{H\to H}\right.$$

$$\left.\left.\left.+\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}R^2\right\|_{H\to H}\right)\left\|R^{N-2-s}\right\|_{H\to H}\right)\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau\right\}+\left\|A^{-\frac{1}{2}}\psi\right\|_H\right]$$

$$+\frac{1}{2}\left[\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}R^2\right\|_{H\to H}\right)\left\|\left(I+i\tau A^{1/2}\right)^{-1}\widehat{R}_{10}\right\|_{H\to H}\left\|R^{N-2}\right\|_{H\to H}\right.$$

$$\left.+\left(\left\|\tau\widehat{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\widehat{R}_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}\widehat{R}^2\right\|_{H\to H}\right)\left\|\left(I+i\tau A^{1/2}\right)^{-1}R_{10}\right\|_{H\to H}\left\|\tilde{R}^{N-2}\right\|_{H\to H}\right]$$

$$\times\left[|\alpha|\frac{1}{2}\left(\left(\left\|\left(I+i\tau A^{1/2}\right)\tilde{R}^N\right\|_{H\to H}+\left\|\left(I+i\tau A^{1/2}\right)R^N\right\|_{H\to H}\right)\left\|\tau A^{1/2}R_3\right\|_{H\to H}\tau\left\|A^{-\frac{1}{2}}f_{1,1}\right\|_H\right.\right.$$

$$+\left\|\tau A^{\frac{1}{2}}R_4\right\|_{H\to H}\sum_{s=1}^{N-1}\left(\left\|\left(I+i\tau A^{1/2}\right)\tilde{R}^{N-s}\right\|_{H\to H}+\left\|\left(I+i\tau A^{1/2}\right)R^{N-s}\right\|_{H\to H}\right)\left\|A^{-\frac{1}{2}}f_s\right\|_H\tau\right)$$

$$+\|(I+i\tau A^{1/2})\varphi\|_H\Big]\Big\}$$

$$\leq M\left\{\sum_{s=1}^{k-1}\left\|A^{-1/2}f_s\right\|_H\tau+\left\|\left(I+i\tau A^{1/2}\right)\varphi\right\|_H+\left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{1,1}\right\|_H\right\} \quad (3.21)$$

kararlılık kestirimleri elde edilir. Aynı adımlarla, (3.15) ve (3.16) formüllerine $A^{\frac{1}{2}}$ operatörü uygulanarak

$$\|A^{1/2}(I+i\tau A^{1/2})\mu\|_H \leq \|T_\tau\|_{H\to H}$$

$$\times\left\{\left[|\alpha|\left(\frac{1}{2}\left(\left\|\left(I+i\tau A^{1/2}\right)\tilde{R}^N\right\|_{H\to H}+\left\|\left(I+i\tau A^{1/2}\right)R^N\right\|_{H\to H}\right)\left\|\tau A^{1/2}R_3\right\|_{H\to H}\tau\left\|f_{1,1}\right\|_H\right.\right.\right.$$

$$\left.+\frac{1}{2}\left\|\tau A^{1/2}R_4\right\|_{H\to H}\sum_{s=1}^{N-1}\left(\left\|\left(I+i\tau A^{1/2}\right)\tilde{R}^{N-s}\right\|_{H\to H}+\left\|\left(I+i\tau A^{1/2}\right)R^{N-s}\right\|_{H\to H}\right)\left\|f_s\right\|_{H\to H}\tau\right)$$

$$+\left\|A^{1/2}\left(I+i\tau A^{1/2}\right)\varphi\right\|_H\Bigg]\Bigg[1+\frac{1}{2}\left(\left(\left\|\tau\widehat{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\widehat{R}_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{1/2}\widehat{R}^2\right\|_{H\to H}\right)\left\|\tilde{R}^{N-2}\right\|_{H\to H}\right.$$

$$\left.\left.+\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\right)\left\|R^{N-2}\right\|_{H\to H}\right)\left\|A^{1/2}R_2\right\|_{H\to H}\right]$$

$$+|\alpha|\frac{1}{2}\left(\|(I+i\tau A^{1/2})\tilde{R}^N\|_{H\to H}+\|(I+i\tau A^{1/2})R^N\|_{H\to H}\right)\|A^{1/2}R_2\|_{H\to H}$$

$$\times\Big[|\beta|\frac{1}{2}\Big\{\Big(\Big(\|\tau\widetilde{R}_7\|_{H\to H}\|\tau^{-1}A^{-1/2}\widetilde{R}_5\|_{H\to H}+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\Big)\|\tilde{R}^{N-2}\|_{H\to H}$$

$$+\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\right)\left\|R^{N-2}\right\|_{H\to H}\Big)\left\|\tau A^{1/2}R_3\right\|_{H\to H}\tau\left\|f_{1,1}\right\|_H$$

$$+\frac{2\tau}{3}\left\|f_N\right\|_H+\left\|\tau R_8\right\|_{H\to H}\left\|f_{N-1}\right\|_H\tau+\left\|\tau R_9\right\|_{H\to H}\left\|f_{N-2}\right\|_H\tau$$

$$+\left\|\tau A^{1/2}R_4\right\|_{H\to H}\sum_{s=1}^{N-3}\left(\left(\left\|\tau\widehat{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\widehat{R}_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{1/2}\widehat{R}^2\right\|_{H\to H}\right)\left\|\widetilde{R}^{N-2-s}\right\|_{H\to H}\right.$$

$$\left.+\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\right)\left\|R^{N-2-s}\right\|_{H\to H}\right)\left\|f_s\right\|_H\tau\Bigg\}+\left\|\psi\right\|_H\Bigg]\Bigg\}$$

$$\leq M\left\{\sum_{s=1}^{k-1}\left\|f_s\right\|_H\tau+\left\|A^{1/2}\left(I+i\tau A^{1/2}\right)\varphi\right\|_H+\left\|\psi\right\|_H+\tau\left\|f_{1,1}\right\|_H\right\} \tag{3.22}$$

ve

$$\left\|\omega\right\|_H\leq\left\|T_\tau\right\|_{H\to H}\left\{\left[1+|\alpha|\frac{1}{2}\left(\left\|\left(I+i\tau A^{1/2}\right)^{-1}\widehat{R}_{10}\right\|_{H\to H}\left\|\left(I+i\tau A^{1/2}\right)R^N\right\|_{H\to H}\right.\right.\right.$$

$$\left.\left.+\left\|(I+i\tau A^{1/2})^{-1}R_{10}\right\|_{H\to H}\left\|(I+i\tau A^{1/2})\widetilde{R}^N\right\|_{H\to H}\right)\right]$$

$$\times\left[|\beta|\frac{1}{2}\left\{\left(\left(\left\|\tau\widetilde{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\widetilde{R}_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}\widetilde{R}^2\right\|_{H\to H}\right)\left\|\tilde{R}^{N-2}\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}R^2\right\|_{H\to H}\right)\left\|R^{N-2}\right\|_{H\to H}\right)$$

$$\times\left\|\tau A^{\frac{1}{2}}R_3\right\|_{H\to H}\left\|f_{1,1}\right\|_H\tau+\frac{2}{3}\left\|f_N\right\|_H\tau+\left\|\tau R_8\right\|_{H\to H}\left\|f_{N-1}\right\|_H\tau+\left\|\tau R_9\right\|_{H\to H}\left\|f_{N-2}\right\|_H\tau$$

$$+\left\|A^{\frac{1}{2}}R_4\right\|_{H\to H}\sum_{s=1}^{N-3}\left(\left(\left\|\tau\widehat{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\widehat{R}_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}\widehat{R}^2\right\|_{H\to H}\right)\right.$$

$$\times \|\tilde{R}^{N-2-s}\|_{H\to H} + \left(\|\tau R_7\|_{H\to H} \left\| \tau^{-1} A^{-\frac{1}{2}} R_5 \right\|_{H\to H} + \left\| \frac{1}{3} \tau A^{\frac{1}{2}} R^2 \right\|_{H\to H} \right)$$

$$\times \left\| R^{N-2-s} \right\|_{H\to H} \Big) \|f_s\|_H \, \tau \Big\} + \|\psi\|_H \Big] + \frac{1}{2} \left[\left(\|\tau R_7\|_{H\to H} \left\| \tau^{-1} A^{-\frac{1}{2}} R_5 \right\|_{H\to H} + \left\| \frac{1}{3} \tau A^{\frac{1}{2}} R^2 \right\|_{H\to H} \right) \right.$$

$$\times \left\| \left(I + i\tau A^{1/2} \right)^{-1} \widehat{R}_{10} \right\|_{H\to H} \left\| R^{N-2} \right\|_{H\to H} + \left(\left\| \tau \widehat{R}_7 \right\|_{H\to H} \left\| \tau^{-1} A^{-\frac{1}{2}} \widehat{R}_5 \right\|_{H\to H} + \left\| \frac{1}{3} \tau A^{\frac{1}{2}} \widehat{R}^2 \right\|_{H\to H} \right)$$

$$\left. \times \left\| \left(I + i\tau A^{1/2} \right)^{-1} R_{10} \right\|_{H\to H} \left\| \tilde{R}^{N-2} \right\|_{H\to H} \right]$$

$$\times \left[|\alpha| \left(\frac{1}{2} \left(\left\| \left(I + i\tau A^{1/2} \right) \tilde{R}^N \right\|_{H\to H} + \left\| \left(I + i\tau A^{1/2} \right) R^N \right\|_{H\to H} \right) \right. \right.$$

$$\times \left\| \tau A^{1/2} R_3 \right\|_{H\to H} \tau \left\| f_{1,1} \right\|_H$$

$$+ \frac{1}{2} \left\| \tau A^{\frac{1}{2}} R_4 \right\|_{H\to H} \sum_{s=1}^{N-1} \left(\left\| \left(I + i\tau A^{1/2} \right) \tilde{R}^{N-s} \right\|_{H\to H} + \left\| \left(I + i\tau A^{1/2} \right) R^{N-s} \right\|_{H\to H} \right) \|f_s\|_H \, \tau \Big)$$

$$\left. \left. + \left\| A^{\frac{1}{2}} (I + i\tau A^{1/2}) \varphi \right\|_H \right] \right\}$$

$$\leq M \left\{ \sum_{s=1}^{k-1} \|f_s\|_H \, \tau + \left\| A^{1/2} \left(I + i\tau A^{1/2} \right) \varphi \right\|_H + \|\psi\|_H + \tau \left\| f_{1,1} \right\|_H \right\} \tag{3.23}$$

kararlılık kestirimleri elde edildi. Daha sonra (3.15) ve (3.16) formüllerine Abel formülü uygulanarak

$$\mu = T_\tau \left\{ \left[\alpha \left(\frac{1}{2} (\tilde{R}^N - R^N) R_3 \tau^2 f_{1,1} + \frac{1}{2} R_4 \tau^2 \left(\sum_{s=2}^{N-1} \left(R_6 R^{N-s} - \widetilde{R}_6 \tilde{R}^{N-s} \right) \right. \right. \right. \right.$$

$$\left. \left. \left. \times (f_s - f_{s-1}) + \left(\widetilde{R}_6 - R_6 \right) f_{N-1} - \left(\widetilde{R}_6 \tilde{R}^{N-1} - R_6 R^{N-1} \right) f_1 \right) \right) + \varphi \right]$$

$$\times\Big[I-\frac{1}{2}\Big(\Big(\widetilde{R}_7\widetilde{R}_5-\frac{\tau A}{3}\widetilde{R}^2\Big)\tilde{R}^{N-2}-\Big(R_7R_5-\frac{\tau A}{3}R^2\Big)R^{N-2}\Big)R_2\Big]$$

$$+\alpha\frac{1}{2}\left(\tilde{R}^N-R^N\right)R_2\left[\beta\left\{\frac{1}{2}\left(\left(\widehat{R}_7\widehat{R}_5-\frac{\tau A}{3}\widehat{R}^2\right)\tilde{R}^{N-2}-\left(R_7R_5-\frac{\tau A}{3}R^2\right)R^{N-2}\right)R_3\tau^2f_{1,1}\right.\right.$$

$$+\frac{\tau}{3}f_N+\frac{1}{2}R_8f_{N-1}\tau^2+\frac{1}{2}R_9f_{N-2}\tau^2+R_4\frac{1}{2}\tau^2\left(\sum_{s=2}^{N-3}\left(R_6\left(R_7R_5-\frac{\tau A}{3}R^2\right)R^{N-2-s}\right.\right.$$

$$-\widetilde{R}_6\Big(\widetilde{R}_7\widetilde{R}_5-\frac{\tau A}{3}\widetilde{R}^2\Big)\tilde{R}^{N-2-s}\Big)(f_s-f_{s-1})$$

$$+\Big(\widetilde{R}_6\Big(\widetilde{R}_7\widetilde{R}_5-\frac{\tau A}{3}\widetilde{R}^2\Big)-R_6\Big(R_7R_5-\frac{\tau A}{3}R^2\Big)\Big)f_{N-3}$$

$$\left.\left.\left.-\left(\widehat{R}_6\left(\widehat{R}_7\widehat{R}_5-\frac{\tau A}{3}\widehat{R}^2\right)\tilde{R}^{N-3}-R_6\left(R_7R_5-\frac{\tau A}{3}R^2\right)R^{N-3}\right)f_1\right)\right\}+\psi\right] \tag{3.24}$$

ve

$$\omega=T_\tau\Big\{\Big[I-\alpha\frac{1}{2}\Big(\widetilde{R}_1R^N-R_1\tilde{R}^N\Big)\Big]$$

$$\times\left[\beta\left\{\frac{1}{2}\left(\left(\widehat{R}_7\widehat{R}_5-\frac{\tau A}{3}\widehat{R}^2\right)\tilde{R}^{N-2}-\left(R_7R_5-\frac{\tau A}{3}R^2\right)R^{N-2}\right)R_3\tau^2f_{1,1}\right.\right.$$

$$+\frac{\tau}{3}f_N+\frac{1}{2}R_8f_{N-1}\tau^2+\frac{1}{2}R_9f_{N-2}\tau^2+R_4\frac{1}{2}\tau^2\left(\sum_{s=2}^{N-3}\left(R_6\left(\frac{R_7R_5}{\tau}-\frac{\tau A}{3}R^2\right)R^{N-2-s}\right.\right.$$

$$-\widehat{R}_6\left(\widehat{R}_7\widehat{R}_5-\frac{\tau A}{3}\widehat{R}^2\right)\tilde{R}^{N-2-s}\Big)\left(f_s-f_{s-1}\right)+\left(\widehat{R}_6\left(\widehat{R}_7\widehat{R}_5-\frac{\tau A}{3}\widehat{R}^2\right)-R_6\left(R_7R_5-\frac{\tau A}{3}R^2\right)\right)f_{N-3}$$

$$-\left(\widetilde{R}_6\left(\widetilde{R}_7\widetilde{R}_5 - \frac{\tau A}{3}\widetilde{R}^2\right)\tilde{R}^{N-3} - R_6\left(R_7R_5 - \frac{\tau A}{3}R^2\right)R^{N-3}\right)f_1\Big)\Big\} + \psi\Big]$$

$$+\frac{1}{2}\left[\left(R_7R_5 - \frac{\tau A}{3}R^2\right)\widetilde{R}_1R^{N-2} - \left(\widetilde{R}_7\widetilde{R}_5 - \frac{\tau A}{3}\widetilde{R}^2\right)R_1\tilde{R}^{N-2}\right]$$

$$\times\Bigg[\alpha\Bigg(\frac{1}{2}(\tilde{R}^N - R^N)R_3\tau^2 f_{1,1} + \frac{1}{2}R_4\tau^2\Bigg(\sum_{s=2}^{N-1}\left(R_6R^{N-s} - \widetilde{R}_6\tilde{R}^{N-s}\right)(f_s - f_{s-1})$$

$$+\left(\widehat{R}_6 - R_6\right)f_{N-1} - \left(\widehat{R}_6\tilde{R}^{N-1} - R_6R^{N-1}\right)f_1\Bigg)\Bigg) + \varphi\Bigg]\Bigg\} \qquad (3.25)$$

formülleri elde edilir. Şimdi, $\|A(I+i\tau A^{1/2})\mu\|_H$ ve $\|A^{1/2}\omega\|_H$ için kararlılık kestirimleri elde edilecek. Öncelikle (3.24) formülüne, A operatörü ve üçgen eşitsizliği uygulanarak ve (2.31), (3.9), (3.10) kestirimlerinden yararlanarak

$$\|A(I+i\tau A^{1/2})\mu\|_H \le \|T_\tau\|_{H\to H}$$

$$\times\Bigg\{\Bigg[|\alpha|\Bigg(\frac{1}{2}\left(\left\|\left(I+i\tau A^{1/2}\right)\tilde{R}^N\right\|_{H\to H} + \left\|\left(I+i\tau A^{1/2}\right)R^N\right\|_{H\to H}\right)\left\|\tau A^{1/2}R_3\right\|_{H\to H}\left\|A^{1/2}f_{1,1}\right\|_H\tau$$

$$+\frac{1}{2}\left\|\tau A^{1/2}R_4\right\|_{H\to H}\Bigg(\sum_{s=2}^{N-1}\Bigg(\left\|\tau A^{1/2}R_6\right\|_{H\to H}\left\|\left(I+i\tau A^{1/2}\right)R^{N-s}\right\|_{H\to H}$$

$$+\left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H}\|(I+i\tau A^{1/2})\tilde{R}^{N-s}\|_{H\to H}\Bigg)\|f_s - f_{s-1}\|_H$$

$$+\left(\left\|\tau A^{1/2}(I+i\tau A^{1/2})\widetilde{R}_6\right\|_{H\to H} + \|\tau A^{1/2}(I+i\tau A^{1/2})R_6\|_{H\to H}\right)\|f_{N-1}\|_H$$

$$+\Big(\left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H}\|(I+i\tau A^{1/2})\tilde{R}^{N-1}\|_{H\to H}$$

$$+\|\tau A^{1/2}R_6\|_{H\to H}\|(I+i\tau A^{1/2})R^{N-1}\|_{H\to H}\big)\|f_1\|_H\big)\big)+\|A(I+i\tau A^{1/2})\varphi\|_H\Big]$$

$$\Big[1+\frac{1}{2}\Big(\Big(\|\tau\widetilde{R}_7\|_{H\to H}\Big\|\tau^{-1}A^{-\frac{1}{2}}\widetilde{R}_5\Big\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{\frac{1}{2}}\widetilde{R}^2\Big\|_{H\to H}\Big)\|\tilde{R}^{N-2}\|_{H\to H}$$

$$+\Big(\|\tau R_7\|_{H\to H}\Big\|\tau^{-1}A^{-\frac{1}{2}}R_5\Big\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{\frac{1}{2}}R^2\Big\|_{H\to H}\Big)\|R^{N-2}\|_{H\to H}\Big)\Big\|A^{\frac{1}{2}}R_2\Big\|_{H\to H}\Big]$$

$$+|\alpha|\frac{1}{2}\big(\|(I+i\tau A^{1/2})\tilde{R}^N\|_{H\to H}+\|(I+i\tau A^{1/2})R^N\|_{H\to H}\big)\|A^{1/2}R_2\|_{H\to H}$$

$$\times\Big[|\beta|\frac{1}{2}\Big\{\Big(\Big(\Big(\|\tau\widetilde{R}_7\|_{H\to H}\Big\|\tau^{-1}A^{-\frac{1}{2}}\widetilde{R}_5\Big\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{\frac{1}{2}}\widetilde{R}^2\Big\|_{H\to H}\Big)\|\tilde{R}^{N-2}\|_{H\to H}$$

$$+\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}R^2\right\|_{H\to H}\right)\left\|R^{N-2}\right\|_{H\to H}\right)\left\|\tau A^{1/2}R_3\right\|_{H\to H}\tau\left\|A^{1/2}f_{1,1}\right\|_H$$

$$+\frac{2\tau}{3}\left\|A^{1/2}f_N\right\|_H+\left\|\tau R_8\right\|_{H\to H}\left\|A^{1/2}f_{N-1}\right\|_H\tau+\left\|\tau R_9\right\|_{H\to H}\left\|A^{1/2}f_{N-2}\right\|_H\tau+\left\|\tau A^{1/2}R_4\right\|_{H\to H}$$

$$\times\left(\sum_{s=2}^{N-3}\left(\left\|\tau A^{1/2}R_6\right\|_{H\to H}\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}R^2\right\|_{H\to H}\right)\left\|R^{N-2-s}\right\|_{H\to H}\right.\right.$$

$$\left.\left.+\left\|\tau A^{1/2}\widehat{R}_6\right\|_{H\to H}\left(\left\|\tau\widehat{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\widehat{R}_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}\widehat{R}^2\right\|_{H\to H}\right)\tilde{R}^{N-2-s}\right)\left\|f_s-f_{s-1}\right\|_H\right.$$

$$+\Big(\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}\Big(\|\tau\widetilde{R}_7\|_{H\to H}\Big\|\tau^{-1}A^{-\frac{1}{2}}\widetilde{R}_5\Big\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{\frac{1}{2}}\widetilde{R}^2\Big\|_{H\to H}\Big)$$

$$+\|\tau A^{1/2}R_6\|_{H\to H}\Big(\|\tau R_7\|_{H\to H}\Big\|\tau^{-1}A^{-\frac{1}{2}}R_5\Big\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{\frac{1}{2}}R^2\Big\|_{H\to H}\Big)\Big)\|f_{N-3}\|_H$$

$$+\Big(\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}\Big(\|\tau\widetilde{R}_7\|_{H\to H}\Big\|\tau^{-1}A^{-\frac{1}{2}}\widetilde{R}_5\Big\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{\frac{1}{2}}\widetilde{R}^2\Big\|_{H\to H}\Big)\|\tilde{R}^{N-3}\|_{H\to H}$$

$$+\|\tau A^{1/2}R_6\|_{H\to H}\Big(\|\tau R_7\|_{H\to H}\Big\|\tau^{-1}A^{-\frac{1}{2}}R_5\Big\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{\frac{1}{2}}R^2\Big\|_{H\to H}\Big)$$

$$\times\|R^{N-3}\|_{H\to H})\|f_1\|_H)\}+\|A^{1/2}\psi\|_H\Big]$$

$$\leq M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\left\|A\left(I+i\tau A^{1/2}\right)\varphi\right\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\} \qquad (3.26)$$

elde edildi. Daha sonra, (3.25) formülüne $A^{1/2}$ operatörü ve üçgen eşitsizliği uygulanarak, (2.31), (3.9), (3.10) kestirimleri yardımı ile

$$\left\|A^{1/2}\omega\right\|_H\leq\|T_\tau\|_{H\to H}\left\{\left[1+|\alpha|\frac{1}{2}\left(\left\|\left(I+i\tau A^{1/2}\right)^{-1}\widehat{R}_{10}\right\|_{H\to H}\left\|\left(I+i\tau A^{1/2}\right)R^N\right\|_{H\to H}\right.\right.\right.$$

$$\left.\left.+\Big\|(I+i\tau A^{1/2})^{-1}R_{10}\Big\|_{H\to H}\Big\|(I+i\tau A^{1/2})\widetilde{R}^N\Big\|_{H\to H}\right)\right]$$

$$\times\Big[|\beta|\frac{1}{2}\Big\{\Big(\Big(\Big(\|\tau\widetilde{R}_7\|_{H\to H}\Big\|\tau^{-1}A^{-\frac{1}{2}}\widetilde{R}_5\Big\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{\frac{1}{2}}\widetilde{R}^2\Big\|_{H\to H}\Big)\|\tilde{R}^{N-2}\|_{H\to H}$$

$$+\Big(\|\tau R_7\|_{H\to H}\Big\|\tau^{-1}A^{-\frac{1}{2}}R_5\Big\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{\frac{1}{2}}R^2\Big\|_{H\to H}\Big)\|R^{N-2}\|_{H\to H}\Big)$$

$$\times\|\tau A^{1/2}R_3\|_{H\to H}\tau\Big\|A^{\frac{1}{2}}f_{1,1}\Big\|_H+\frac{2}{3}\tau\|A^{1/2}f_N\|_H+\|\tau R_8\|_{H\to H}\|A^{1/2}f_{N-1}\|_H\tau$$

$$+\|\tau R_9\|_{H\to H}\|A^{1/2}f_{N-2}\|_H\tau+\|\tau A^{1/2}R_4\|_{H\to H}\tau\left(\sum_{s=2}^{N-3}\Big(\|\tau A^{1/2}R_6\|_{H\to H}\right.$$

$$\left(\|\tau R_7\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}R^2\right\|_{H\to H}\right)\|R^{N-2-s}\|_{H\to H}$$

$$+\left\|\tau A^{1/2}\widehat{R}_6\right\|_{H\to H}\left(\left\|\tau\widehat{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\widehat{R}_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}\widehat{R}^2\right\|_{H\to H}\right)\left\|\tilde{R}^{N-2-s}\right\|_{H\to H}\Big)\|f_s-f_{s-1}\|_H$$

$$+\left(\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}\left(\|\tau\widetilde{R}_7\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\widetilde{R}_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}\widetilde{R}^2\right\|_{H\to H}\right)\right.$$

$$\left.+\|\tau A^{1/2}R_6\|_{H\to H}\left(\|\tau R_7\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}R^2\right\|_{H\to H}\right)\right)\|f_{N-3}\|_H$$

$$+\left(\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}\left(\|\tau\widetilde{R}_7\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\widetilde{R}_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}\widetilde{R}^2\right\|_{H\to H}\right)\|\tilde{R}^{N-3}\|_{H\to H}\right.$$

$$\left.+\left\|\tau A^{1/2}R_6\right\|_{H\to H}\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}R^2\right\|_{H\to H}\right)\left\|R^{N-3}\right\|_{H\to H}\right)$$

$$\times\left\|f_1\right\|_H\Big)\Big\}+\left\|A^{1/2}\psi\right\|_H\Big]+\frac{1}{2}\Bigg[\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}R_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}R^2\right\|_{H\to H}\right)$$

$$\times\left\|\widetilde{R}_{10}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|(I+i\tau A^{1/2})R^{N-2}\|_{H\to H}$$

$$+\left(\|\tau\widetilde{R}_7\|_{H\to H}\left\|\tau^{-1}A^{-\frac{1}{2}}\widetilde{R}_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{\frac{1}{2}}\widetilde{R}^2\right\|_{H\to H}\right)$$

$$\times\left\|R_{10}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|(I+i\tau A^{1/2})\tilde{R}^{N-2}\|_{H\to H}\Bigg]$$

$$\times\Bigg[|\alpha|\frac{1}{2}\Big(\Big(\left\|\tilde{R}^N\right\|_{H\to H}+\left\|R^N\right\|_{H\to H}\Big)\left\|\tau A^{1/2}R_3\right\|_{H\to H}\tau\left\|A^{1/2}f_{1,1}\right\|_H$$

$$+\left\|\tau A^{1/2}R_4\right\|_{H\to H}\left(\sum_{s=2}^{N-1}\left(\left\|\tau A^{1/2}R_6\right\|_{H\to H}\left\|R^{N-s}\right\|_{H\to H}+\left\|\tau A^{1/2}\widehat{R}_6\right\|_{H\to H}\left\|\tilde{R}^{N-s}\right\|_{H\to H}\right)\right.$$

$$\times\|f_s-f_{s-1}\|_H+\left(\left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H}+\|\tau A^{1/2}R_6\|_{H\to H}\right)\|f_{N-1}\|_{H\to H}$$

$$\left.+\left(\left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H}\|\tilde{R}^{N-1}\|_{H\to H}+\|\tau A^{1/2}R_6\|_{H\to H}\|R^{N-1}\|_{H\to H}\right)\|f_1\|_H\right)$$

$$\left.+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|A(I+i\tau A^{1/2})\varphi\|_H\right]\Bigg\}$$

$$\leq M\left\{\sum_{s=2}^{N-1}\left\|f_s-f_{s-1}\right\|_H+\left\|f_1\right\|_H+\left\|A\left(I+i\tau A^{1/2}\right)\varphi\right\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}\qquad(3.27)$$

kararlılık kestirimi elde edilir. Şimdi, sırasıyla (3.17), (3.18), (3.19) kestirimleri elde edilecek. İlk olarak, (3.12) formülü ve (2.31), (3.9), (3.20), (3.21), (3.22), (3.23) kestirimleri kullanılarak, üçgen eşitsizliği uygulanırsa $k\geq 2$ hali için

$$\|u_k\|_H\leq\frac{1}{2}\left(\left\|\widetilde{R}_{10}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|R^k\|_{H\to H}\right.$$

$$\left.+\left\|R_{10}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|\tilde{R}^k\|_{H\to H}\right)\|(I+i\tau A^{1/2})\mu\|_H$$

$$+\frac{1}{2}\left(\|A^{1/2}R_2\|_{H\to H}\|\tilde{R}^k\|_{H\to H}+\|A^{1/2}R_2\|_{H\to H}\|R^k\|_{H\to H}\right)\|A^{-1/2}\omega\|_H$$

$$+\frac{1}{2}\left(\|\tau A^{1/2}R_3\|_{H\to H}\|\tilde{R}^k\|_{H\to H}+\|\tau A^{1/2}R_3\|_{H\to H}\|R^k\|_{H\to H}\right)\tau\|A^{-1/2}f_{1,1}\|_H$$

$$+\frac{1}{2}\|\tau A^{1/2}R_4\|_{H\to H}\sum_{s=1}^{k-1}\left[\|\tilde{R}^{k-s}\|_{H\to H}+\|R^{k-s}\|_{H\to H}\right]\|A^{-1/2}f_s\|_H\tau$$

$$\leq M\left\{\sum_{s=1}^{N-1}\left\|A^{-1/2}f_s\right\|_H\tau+\left\|\left(I+i\tau A^{1/2}\right)\varphi\right\|_H+\left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{1,1}\right\|_H\right\}$$

kararlılık kestirimi elde edilir. Daha sonra (3.12) formülüne $A^{1/2}$ operatörü uygulanarak, $k\geq 2$ hali için

$$\|A^{1/2}u_k\|_H \leq \frac{1}{2}\Big(\left\|\widetilde{R}_{10}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|R^k\|_{H\to H}$$

$$+\left\|R_{10}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|\tilde{R}^k\|_{H\to H}\Big)\|A^{1/2}(I+i\tau A^{1/2})\mu\|_H$$

$$+\frac{1}{2}\left(\|A^{1/2}R_2\|_{H\to H}\|\tilde{R}^k\|_{H\to H}+\|A^{1/2}R_2\|_{H\to H}\|R^k\|_{H\to H}\right)\|\omega\|_H$$

$$+\frac{1}{2}\left(\|\tau A^{1/2}R_3\|_{H\to H}\|\tilde{R}^k\|_{H\to H}+\|\tau A^{1/2}R_3\|_{H\to H}\|R^k\|_{H\to H}\right)\tau\|f_{1,1}\|_H$$

$$+\frac{1}{2}\|\tau A^{1/2}R_4\|_{H\to H}\sum_{s=1}^{k-1}\left(\|\tilde{R}^{k-s}\|_{H\to H}+\|R^{k-s}\|_{H\to H}\right)\|f_s\|_H\tau$$

$$\leq M\left\{\sum_{s=1}^{N-1}\left\|f_s\right\|_H\tau+\left\|A^{1/2}\left(I+i\tau A^{1/2}\right)\varphi\right\|_H+\left\|\psi\right\|_H+\tau\left\|f_{1,1}\right\|_H\right\}$$

kararlılık kestirimi elde edilir. (3.12) ye Abel formülü uygulanarak

$$u_k = \frac{1}{2}\left[\widetilde{R}_1R^k-R_1\tilde{R}^k\right]\mu+\frac{1}{2}[\tilde{R}^k-R^k]R_2\omega+\frac{1}{2}[\tilde{R}^k-R^k]R_3\tau^2f_{1,1}$$

$$+\tau^2R_4\frac{1}{2}\Bigg(\sum_{s=2}^{k-1}\left[R_6R^{k-s}-\widetilde{R}_6\tilde{R}^{k-s}\right](f_s-f_{s-1})$$

$$+\left(\widehat{R}_6-R_6\right)f_{k-1}-\left[\widehat{R}_6\tilde{R}^{k-1}-R_6R^{k-1}\right]f_1\Bigg),\ 2\leq k\leq N \qquad (3.28)$$

formülü elde edildi. Son adımda (3.28) e A operatörü ve üçgen eşitsizliği uygulanarak ve (2.31), (3.9) kestirimlerinden yararlanarak, $k \geq 2$ hali için

$$\|Au_k\|_H \leq \frac{1}{2}\left(\left\|\widetilde{R}_{10}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|R^k\|_{H\to H} + \left\|R_{10}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|\tilde{R}^k\|_{H\to H}\right)$$

$$\times \|A(I+i\tau A^{1/2})\mu\|_H + \frac{1}{2}\left(\|A^{1/2}R_2\|_{H\to H}\|\tilde{R}^k\|_{H\to H} + \|A^{1/2}R_2\|_{H\to H}\|R^k\|_{H\to H}\right)\|A^{1/2}\omega\|_H$$

$$+ \frac{1}{2}\left(\|\tau A^{-1/2}R_3\|_{H\to H}\|\tilde{R}^k\|_{H\to H} + \|\tau A^{-1/2}R_3\|_{H\to H}\|R^k\|_{H\to H}\right)\tau\|A^{1/2}f_{1,1}\|_H$$

$$+ \frac{1}{2}\|\tau A^{1/2}R_4\|_{H\to H}\left(\sum_{s=2}^{k-1}\left[\|\tau A^{1/2}R_6\|_{H\to H}\|R^{k-s}\|_{H\to H} + \left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H}\|\tilde{R}^{k-s}\|_{H\to H}\right]\right.$$

$$\times\| f_s - f_{s-1} \|_H + \left(\left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H} + \|\tau A^{1/2}R_6\|_{H\to H}\right) \| f_{k-1} \|_H$$

$$\left.+\left[\left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H} + \|\tau A^{1/2}R_6\|_{H\to H}\|R^{k-1}\|_{H\to H}\right] \| f_1 \|_H \right)$$

$$\leq M\left\{\sum_{s=2}^{N-1}\|f_s - f_{s-1}\|_H + \|f_1\|_H + \left\|A\left(I+i\tau A^{1/2}\right)\varphi\right\|_H + \left\|A^{1/2}\psi\right\|_H + \tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}$$

kararlılık kestirimi elde edilir. Böylece Teorem 3.1 in ispatı tamamlanmış olur. Şimdi Teorem 3.1 in bazı sonuç teoremlerini verelim. İlk olarak karışık lokal olmayan hiperbolik tip problem

$$\begin{cases} u_{tt} - (a(x)u_x)_x + \delta u = f(t,x), \ 0 < t < 1, \ 0 < x < 1, \\ u(0,x) = \alpha u(1,x) + \varphi(x), 0 \leq x \leq 1, \\ u_t(0,x) = \beta u_t(1,x) + \psi(x), 0 \leq x \leq 1, \\ u(t,0) = u(t,1), \ u_x(t,0) = u_x(t,1), \ 0 \leq t \leq 1 \end{cases} \tag{3.29}$$

ele alındı. Problem (3.29) un bir tek ve düzgün çözümü $u(t,x)$ ($\delta > 0$) mevcuttur ve

$a(x) \geq a > 0$, $(x \in (0,1))$, $\varphi(x), \psi(x) \quad (x \in [0,1])$, $f(t,x)$, $(t, x \in [0,1])$ fonksiyonları da düzgün fonksiyonlardır (Ashyralyev ve Aggez 2004). Bu sayede, Hilbert uzayı $H = L_2[0,1]$ üzerinde, (3.29) da verilen, kendine eşlenik, pozitif tanımlı A^x operatörü yardımı ile, karışık tip problem (3.29), lokal olmayan sınır değer problemi (3.1) e indirgenebilir.

Problem (3.29) un ayrışımı iki adımda ele alındı. İlk adımda grid uzay $[0,1]_h = \{x : x_r = rh, 0 \leq r \leq K, Kh = 1\}$ şeklinde tanımlandı. $[0,1]_h$ aralığında, $\varphi^h(x) = \{\varphi_r\}_1^{K-1}$ grid fonksiyonlarının tanımlı olduğu, Hilbert uzayı $L_{2h} = L_2([0,1]_h)$ ve Sobolev uzayları $W_{2h}^1 = W_{2h}^1([0,1]_h)$, $W_{2h}^2 = W_{2h}^2([0,1]_h)$ verildi. Problem (3.29) daki diferansiyel operatör yerine, (2.65) de verilen fark operatörü A_h^x yazıldı. A_h^x operatörünün yardımıyla, sonsuz çoklukta adi diferansiyel denklem sistemi için lokal olmayan sınır değer problemi

$$\begin{cases} \frac{d^2 v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = f^h(t,x),\ 0 < t < 1,\ x \in (0,1)_h, \\ v^h(0,x) = \alpha v^h(1,x) + \varphi^h(x), x \in [0,1]_h, \\ v_t^h(0,x) = \beta v_t^h(1,x) + \psi^h(x), x \in [0,1]_h \end{cases} \tag{3.30}$$

elde edilir. İkinci adımda (3.30) probleminin fark şeması (3.31) yazılır.

$$\begin{cases} \tau^{-2}\left(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)\right) + \frac{2}{3} A_h^x u_k^h(x) \\ + \frac{1}{6} A_h^x \left(u_{k+1}^h(x) + u_{k-1}^h(x)\right) + \frac{1}{12}\tau^2 \left(A_h^x\right)^2 u_{k+1}^h(x) = f_k^h(x), \\ f_k^h(x) = \frac{2}{3} f^h(t_k, x) + \frac{1}{6}\left(f^h(t_{k+1}, x) + f^h(t_{k-1}, x)\right) \\ - \frac{1}{12}\tau^2 \left(-Af^h(t_{k+1}, x) + f_{tt}^h(t_{k+1}, x)\right), x \in [0,1]_h, \\ t_k = k\tau, N\tau = 1, 1 \leq k \leq N-1, \\ u_0^h(x) = \alpha u_N^h(x) + \varphi^h(x), x \in [0,1]_h, \\ \left(I + \frac{\tau^2}{12}\left(A_h^x\right) + \frac{\tau^4}{144}\left(A_h^x\right)^2\right)\tau^{-1}\left(u_1^h(x) - u_0^h(x)\right) \\ + \frac{\tau}{2}\left(A_h^x\right)\varphi^h(x) - \tau f_{1,1}^h(x) = \beta\left(I - \frac{\tau^2}{12}\left(A_h^x\right)\right) \\ \times\left(\frac{1}{6\tau}\left(7u_N^h(x) - 8u_{N-1}^h(x) + u_{N-2}^h(x)\right) + \frac{\tau}{3}\left(f_N^h(x) - Au_N^h(x)\right)\right) \\ + \left(I - \frac{\tau^2}{12}\left(A_h^x\right)\right)\psi^h(x), x \in [0,1]_h, \\ f_{1,1}^h(x) = \frac{1}{2} f^h(0,x) + \frac{\tau}{6} f_t^h(0,x). \end{cases} \tag{3.31}$$

Teorem 3.2 τ ve h yeterince küçük sayılar olsun. Bu takdirde fark şeması (3.30) un çözümü için aşağıdaki kararlılık kestirimleri sağlanır;

$$\max_{0\le k\le N}\| u_k^h \|_{L_{2h}} + \max_{0\le k\le N}\| u_k^h \|_{W_{2h}^1}$$

$$\le M_1\left[\max_{1\le k\le N-1}\| f_k^h \|_{L_{2h}} + \| \psi^h \|_{L_{2h}} + \| \varphi^h \|_{W_{2h}^1} + \tau \| \varphi^h \|_{W_{2h}^2} + \tau \| f_{1,1}^h \|_{L_{2h}} \right],$$

$$\max_{1\le k\le N-1}\| \tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h) \|_{L_{2h}} + \max_{0\le k\le N}\| u_k^h \|_{W_{2h}^2}$$

$$\le M_1\left[\left\| f_1^h \right\|_{L_{2h}} + \max_{2\le k\le N-1}\left\| \tau^{-1}\left(f_k^h - f_{k-1}^h \right) \right\|_{L_{2h}} + \left\| \psi^h \right\|_{W_{2h}^1} + \left\| \varphi^h \right\|_{W_{2h}^2} + \tau \left\| \varphi^h \right\|_{W_{2h}^3} + \tau \left\| f_{1,1}^h \right\|_{W_{2h}^1} \right].$$

Burada M_1 sabiti, $\tau,\ h,\ \varphi^h(x),\ \psi^h(x),\ f_{1,1}^h$ ve f_k^h $(1\le k<N)$ ifadelerinden bağımsızdır.

Teorem 3.2 nin ispatı soyut (abstract) Teorem 3.1 in ispatına ve (2.65) te verilen A_h^x fark operatörünün simetri özeliğine bağlıdır.

İkinci olarak Ω, m-boyutlu $\mathbb{R}^m\left\{x=(x_1,\cdots,x_m):0<x_j<1, 1\le j\le m\right\}$ Öklid uzayında, sınırları S, $\overline{\Omega}=\Omega\cup S$ olan birim açık küp olsun ve $[0,1]\times\Omega$ de çok boyutlu, karışık tip lokal olmayan hiperbolik problemini

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^{m}(a_r(x)u_{x_r})_{x_r} = f(t,x), \\ x=(x_1,\ldots,x_m)\in\Omega,\ 0<t<1, \\ u^h(0,x)=\alpha u^h\left(1,x\right)+\varphi^h(x), x\in\overline{\Omega}, \\ u_t^h(0,x)=\beta u_t^h(1,x)+\psi^h(x), x\in\overline{\Omega}, \\ u(t,x)=0,\ x\in S \end{cases} \tag{3.32}$$

ele alalım. Burada $a_r(x)$, $(x \in \Omega)$, $\varphi(x)$, $\psi(x)$ $(x \in \overline{\Omega})$, $f(t,x)$ $(t \in (0,1),\ x \in \Omega)$ düzgün fonksiyonlar ve $a_r(x) \geq a > 0$ dır.

$\overline{\Omega}$ da karesi integrallenebilen tüm fonksiyonların tanımlandığı Hilbert uzayı $L_2(\overline{\Omega})$ de, $\| f \|_{L_2(\overline{\Omega})} = \{ \int \cdots \int_{x \in \overline{\Omega}} | f(x) |^2 dx_1 \cdots dx_m \}^{\frac{1}{2}}$ normu tanımlıdır.

Problem (3.33) in çözümü $u(t,x)$ bir tektir ve düzgün fonksiyondur, ayrıca $\varphi(x)$, $\psi(x)$, $a_r(x)$, $f(t,x)$ fonksiyonları da düzgün fonksiyonlardır. Bu sayede, Hilbert uzayı $H = L_2[0,1]$ üzerinde, (3.32) de verilen, kendine eşlenik, pozitif tanımlı A^x operatörü yardımı ile, karışık tip problem (3.32) lokal olmayan sınır değer problemi (3.1) e indirgenebilir.

Problem (3.32) nin ayrışımı iki adımda yapılabilir. Birinci adımda grid uzayı

$$\widetilde{\Omega}_h = \{x = x_r = (h_1 r_1, \cdots, h_m r_m), r = (r_1, \cdots, r_m),$$

$$0 \leq r_j \leq N_j, h_j N_j = 1, j = 1, \cdots, m\}, \Omega_h = \widehat{\Omega}_h \cap \Omega, S_h = \widehat{\Omega}_h \cap S$$

tanımlandı. $\widehat{\Omega}_h$ de $\varphi^h(x) = \{\varphi(h_1 r_1, \cdots, h_m r_m)\}$ grid fonksiyonlarının tanımlı olduğu, Banach uzayı $L_{2h} = L_2(\widehat{\Omega}_h)$ de ve Sobolev uzayı $W_{2h}^1 = W_{2h}^1\left(\widetilde{\Omega}_h\right)$ verilerek, problem (3.32) deki diferansiyel operatör (2.69) verilen A_h^x fark operatörü ile değiştirildi. A_h^x operatörünün yardımıyla sonsuz çoklukta adi diferansiyel denklem sistemi için lokal olmayan sınır değer problemi

$$\begin{cases} \frac{d^2 v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = f^h(t,x),\ 0 < t < 1,\ x \in \Omega_h, \\ v^h(0,x) = \alpha v^h\left(1,x\right) + \varphi^h(x), x \in \widehat{\Omega}_h, \\ \frac{dv^h(0,x)}{dt} = \beta v_t^h(1,x) + \psi^h(x), x \in \widehat{\Omega}_h \end{cases} \tag{3.33}$$

elde edilir. İkinci adımda problem (3.33) ün fark şeması

$$\begin{cases}\tau^{-2}\left(u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)\right)+\frac{2}{3}A_h^x u_k^h(x)\\ +\frac{1}{6}A_h^x\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)+\frac{1}{12}\tau^2\left(A_h^x\right)^2u_{k+1}^h(x)=f_k^h(x),\\ f_k^h(x)=\frac{2}{3}f^h(t_k,x)+\frac{1}{6}\left(f^h(t_{k+1},x)+f^h(t_{k-1},x)\right)\\ -\frac{1}{12}\tau^2\left(-Af^h(t_{k+1},x)+f_{tt}^h(t_{k+1},x)\right),x\in\Omega_h,\\ t_k=k\tau,1\le k\le N-1,N\tau=1,\\ u_0^h(x)=\alpha u_N^h(x)+\varphi^h(x),x\in\widehat{\Omega}_h,\\ \left(I+\frac{\tau^2}{12}\left(A_h^x\right)+\frac{\tau^4}{144}\left(A_h^x\right)^2\right)\tau^{-1}\left(u_1^h(x)-u_0^h(x)\right)\\ +\frac{\tau}{2}\left(A_h^x\right)\varphi^h(x)-\tau f_{1,1}^h(x)=\beta\left(I-\frac{\tau^2}{12}\left(A_h^x\right)\right)\\ \times\left(\dfrac{1}{6\tau}\left(7u_N^h(x)-8u_{N-1}^h(x)+u_{N-2}^h(x)\right)+\frac{\tau}{3}\left(f_N^h(x)-Au_N^h(x)\right)\right)\\ +\left(I-\frac{\tau^2}{12}\left(A_h^x\right)\right)\psi^h(x),x\in\widehat{\Omega}_h,\\ f_{1,1}^h(x)=\frac{1}{2}f^h(0,x)+\frac{\tau}{6}f_t^h(0,x)\end{cases} \tag{3.34}$$

oluşturulur.

Teorem 3.3 τ ve $|h|$ yeterince küçük sayılar olsun. Bu takdirde fark şeması (3.34) ün çözümü için aşağıdaki kararlılık kestirimlerini sağlanır;

$$\max_{0\le k\le N}\|u_k^h\|_{L_{2h}}+\max_{0\le k\le N}\|u_k^h\|_{W_{2h}^1}$$

$$\le M_1\left[\max_{1\le k\le N-1}\|f_k^h\|_{L_{2h}}+\|\psi^h\|_{L_{2h}}+\|\varphi^h\|_{W_{2h}^1}+\tau\|\varphi^h\|_{W_{2h}^2}+\tau\|f_{1,1}^h\|_{L_{2h}}\right],$$

$$\max_{1\le k\le N-1}\|\tau^{-2}(u_{k+1}^h-2u_k^h+u_{k-1}^h)\|_{L_{2h}}+\max_{0\le k\le N}\|u_k^h\|_{W_{2h}^2}$$

$$\le M_1\Big[\|f_1^h\|_{L_{2h}}+\max_{2\le k\le N-1}\|\tau^{-1}(f_k^h-f_{k-1}^h)\|_{L_{2h}}$$

$$+\|\psi^h\|_{W_{2h}^1}+\|\varphi^h\|_{W_{2h}^2}+\tau\|\varphi^h\|_{W_{2h}^3}+\tau\|f_{1,1}^h\|_{W_{2h}^1}\Big].$$

Burada M_1 sabiti, $\tau,\ h,\ \varphi^h(x),\ \psi^h(x), f_{1,1}^h$ ve f_k^h $(1\le k<N)$ ifadelerinden bağımsızdır.

Teorem 3.3 ün ispatı abstract (soyut) Teorem 3.1 ispatına, (2.69) da verilen A_h^x fark operatörün simetri özeliğine ve Teorem 2.4 e bağlıdır.

3.3 Lokal Olmayan Hiperbolik Tip Sınır Değer Problemi İçin Dördüncü Mertebeden Kararlı Doğruluk Fark Şeması

Bu kısımda, lokal olmayan sınır değer problemi (3.1) in nümerik (sayısal) çözümü için dördüncü mertebeden kararlı doğruluk fark şeması ele alındı. Problem (3.1) in çözümü için, fark şeması (2.7) den ve formül (2.20) den yararlanarak, dördüncü mertebeden iki adımlı fark şeması

$$\begin{cases}\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{5}{6}Au_k+\frac{1}{12}A\left(u_{k+1}+u_{k-1}\right)\\ -\frac{\tau^2}{72}A^2u_k+\frac{\tau^2}{144}A^2\left(u_{k+1}+u_{k-1}\right)=f_k,\\ f_k=\frac{5}{6}f(t_k)+\frac{1}{12}\left(f(t_{k+1})+f(t_{k-1})\right)+\frac{\tau^2}{72}\left(-Af(t_k)+f''(t_k)\right)\\ -\frac{1}{144}\tau^2\left(-A\left(f(t_{k+1})+f(t_{k-1})\right)+f''(t_{k+1})+f''(t_{k-1})\right),\\ t_k=k\tau,1\le k\le N-1,N\tau=1,\\ u_0=\alpha u_N+\varphi,\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\tau^{-1}\left(u_1-u_0\right)+\frac{\tau}{2}Au_0-\tau f_{2,2}\\ =\beta\left(I-\frac{\tau^2A}{12}\right)\left(\dfrac{85u_N-108u_{N-1}+27u_{N-2}-4u_{N-3}}{66\tau}+\frac{3\tau}{11}\left(f_N-Au_N\right)\right)\\ +\left(I-\frac{\tau^2A}{12}\right)\psi,\end{cases}\tag{3.35}$$

elde edilir. Burada

$$f_{2,2}=\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\tau^{-1}f_{1,1}^4$$

$$= \left\{ \left(I - \frac{\tau^2 A}{12} \right) f(0) + \left(-\left(I - \frac{5\tau^2 A}{12} \right) f(0) + \tau f'(0) \right) \frac{1}{2} \right.$$

$$\left. + \left(-A\tau f(0) - 2f'(0) + \tau f''(0) \right) \frac{\tau}{6} + \left(Af(0) - 3f''(0) \right) \frac{\tau^2}{24} \right\}$$

tür. Fark şeması (3.35) in çözümünün kararlılığı

$$|\alpha| + \frac{173}{66}|\beta| + \frac{173}{66}|\alpha||\beta| < 1 \tag{3.36}$$

koşulu altında incelendi. Bu kısımda da, kolaylık olması açısından

$$B_n^\tau = -\beta \frac{1}{2} \left[\left(-\frac{3\tau A}{11} \widetilde{R} + \frac{85\widetilde{R} - 108 + 27\widetilde{R}^{-1} - 4\widetilde{R}^{-2}}{66\tau} \right) \widetilde{J}_1 \widetilde{R}^{N-2} \right.$$

$$\left. - \left(-\frac{3\tau A}{11} R + \frac{85R - 108 + 27R^{-1} - 4R^{-2}}{66\tau} \right) J_1 R^{N-2} \right] iA^{-1/2}$$

$$- \alpha \frac{1}{2} \left[J_1 R^{N-1} + \widetilde{J}_1 \widetilde{R}^{N-1} \right]$$

$$+ \alpha\beta \frac{1}{2} iA^{-1/2} J_1 \widetilde{J}_1 \left(-\frac{3\tau A}{11} \widetilde{R} + \frac{85\widetilde{R} - 108 + 27\widetilde{R}^{-1} - 4\widetilde{R}^{-2}}{66\tau} \right) R^{N-1} \widetilde{R}^{N-2}$$

$$-\alpha\beta \frac{1}{2} iA^{-1/2} J_1 \widehat{J}_1 \left(-\frac{3\tau A}{11} R + \frac{85R - 108 + 27R^{-1} - 4R^{-2}}{66\tau} \right) \widehat{R}^{N-1} R^{N-2}$$

eşitliği kullanılacak. Önce, teoremin ispatında yararlanılacak iki lemmayı ele alalım. Bu kısımda ayrıca, Lemma 2.2 de ispat edilen kestirimlerden, $R, \widetilde{R}, J_1, J_2, J_3, J_4$ operatörleri ve bunların eşleniklerinden ve

$$J_7 = \frac{85R - 108 + 27R^{-1} - 4R^{-2}}{66\tau}$$

$$=\frac{-155}{2}\left(I-\frac{3\,704\tau^2A}{1\,860}+\frac{42\,745\tau^4A^2}{22\,320}-\frac{157\tau^6A^3}{267\,840}\right.$$

$$\left.+\frac{135i\tau A^{1/2}}{155}+\frac{3\,483i\tau^3A^{3/2}}{11\,160}+\frac{231i\tau^5A^{5/2}}{22\,320}\right)$$

$$\times\left(66\left(I+\tau^2A+\frac{3\tau^4A^2}{144}+\frac{\tau^6A^3}{864}+\frac{\tau^8A^4}{20736}\right)\right)^{-1} \quad (3.37)$$

ve eşleniği $\tilde{J}_7$

$$\widehat{J}_7=\frac{85\widehat{R}-108+27\widehat{R}^{-1}-4\widehat{R}^{-2}}{66\tau} \quad (3.38)$$

operatörlerinden yararlanıldı.

Lemma 3.3 Aşağıdaki kararlılık kestirimleri sağlanır;

$$\begin{cases}\left\|A^{-1/2}J_7\right\|_{H\to H}\le\frac{155}{66},\left\|A^{-1/2}\widehat{J}_7\right\|_{H\to H}\le\frac{155}{66},\\[2ex] \left\|\tau A^{1/2}\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\le 1.\end{cases} \quad (3.39)$$

İspat: Lemmanın ispatında, (2.73) de verilen kestirimlerinden yararlanılarak Lemma 3.1 in ispatındaki adımlar takip edildi. Kendine eşlenik, pozitif tanımlı operatörlerin spektral özeliği yardımı ile

$$\left\|\frac{-155}{2}iA^{1/2}\left(I-\frac{3\,704\tau^2A}{1\,860}+\frac{42\,745\tau^4A^2}{22\,320}-\frac{157\tau^6A^3}{267\,840}\right.\right.$$

$$\left.+\frac{135i\tau A^{1/2}}{155}+\frac{3\,483i\tau^3A^{3/2}}{11\,160}+\frac{231i\tau^5A^{5/2}}{22\,320}\right)$$

$$\left.\times\left(66A^{1/2}\left(I+\tau^2A+\frac{3\tau^4A^2}{144}+\frac{\tau^6A^3}{864}+\frac{\tau^8A^4}{20\,736}\right)\right)^{-1}\right\|_{H\to H}$$

$$\leq \sup_{\delta \leq \lambda < \infty} \left| \left(\frac{155}{132} \right) \frac{1 - \frac{37\,04\tau^2\lambda}{1\,860} + \frac{42\,745\tau^4\lambda^2}{22\,320} - \frac{157\tau^6\lambda^3}{267\,840} + \frac{135i\tau\lambda^{1/2}}{155} + \frac{3\,483i\tau^3\lambda^{3/2}}{11\,160} + \frac{231i\tau^5\lambda^{5/2}}{22\,320}}{1 + \tau^2\lambda + \frac{3\tau^4\lambda^2}{144} + \frac{\tau^6\lambda^3}{864} + \frac{\tau^8\lambda^4}{20\,736}} \right|$$

yazılabilir. Buradan da

$$\frac{\left(1 - \frac{3\,704\tau^2\lambda}{1\,860} + \frac{42\,745\tau^4\lambda^2}{22\,320} - \frac{157\tau^6\lambda^3}{267\,840}\right)^2 + \left(\frac{135\tau\lambda^{1/2}}{155} + \frac{3\,483\tau^3\lambda^{3/2}}{11\,160} + \frac{231\tau^5\lambda^{5/2}}{22\,320}\right)^2}{\left(1 + \tau^2\lambda + \frac{3\tau^4\lambda^2}{144} + \frac{\tau^6\lambda^3}{864} + \frac{\tau^8\lambda^4}{20\,736}\right)^2} \leq 2$$

olduğundan, $\left\| A^{-1/2} J_7 \right\|_{H \to H} \leq \dfrac{155}{66}$, $\left\| A^{-1/2} \widehat{J}_7 \right\|_{H \to H} \leq \dfrac{155}{66}$ dır. Aynı şekilde

$$\left\| \tau A^{1/2} \left(I + i\tau A^{1/2} \right)^{-1} \right\|_{H \to H} \leq \sup_{\delta \leq \lambda < \infty} \left| \frac{\tau\lambda^{1/2}}{1 + i\tau\lambda^{1/2}} \right|$$

yazılabilir. Buradan da

$$\frac{\tau^2\lambda}{1 + \tau^2\lambda} \leq 1,$$

olduğundan, $\left\| \tau A^{1/2} \left(I + i\tau A^{1/2} \right)^{-1} \right\|_{H \to H} \leq 1$ dir. Bu, Lemma 3.3 ü ispat eder.

Lemma 3.4 Koşul (3.36) sağlansın, bu takdirde $I - B_n^\tau$ operatörünün sınırlı bir tersi $T_\tau = \left(I - B_n^\tau \right)^{-1}$ mevcuttur. *A* operatörünün simetri özeliği ve pozitif tanımlılığı yardımı ile

$$\left\| T_\tau \right\|_{H \mapsto H} \leq \frac{1}{1 - |\alpha| - \frac{173}{66}|\beta| - \frac{173}{66}|\alpha||\beta|} \tag{3.40}$$

eşitsizliği yazılabilir.

İspat: B_n^τ, R, $\tilde{R}$ operatörlerinden ve (2.73), (3.39) kestirimlerinden yararlanarak, üçgen eşitsizliği uygulanırsa,

$$B_n^\tau \leq |\beta|\frac{1}{2}\Big(\Big(\frac{3}{11}\left\|\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|\tilde{R}\|_{H\to H}$$

$$+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|A^{-1/2}\tilde{J}_7\right\|_{H\to H}\Big)\|\tilde{J}_1\|_{H\to H}\|\tilde{R}^{N-2}\|_{H\to H}$$

$$+\Big(\frac{3}{11}\left\|\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|R\|_{H\to H}+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|A^{-1/2}J_7\|_{H\to H}\Big)$$

$$\times\|J_1\|_{H\to H})+|\alpha|\frac{1}{2}\Big[\|J_1\|_{H\to H}\|R^{N-1}\|_{H\to H}+\left\|\tilde{J}_1\right\|_{H\to H}\|\tilde{R}^{N-1}\|_{H\to H}\Big]$$

$$+|\alpha||\beta|\frac{1}{2}\Big(\frac{3}{11}\left\|\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|\tilde{R}\|_{H\to H}$$

$$+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|A^{-1/2}\tilde{J}_7\right\|_{H\to H}\Big)\|J_1\|_{H\to H}\left\|\tilde{J}_1\right\|_{H\to H}\|R^{N-1}\|_{H\to H}\left\|\tilde{R}^{N-2}\right\|_{H\to H}$$

$$+|\alpha||\beta|\frac{1}{2}\Big(\frac{3}{11}\left\|\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|R\|_{H\to H}$$

$$+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|A^{-1/2}J_7\|_{H\to H}\Big)$$

$$\times\|J_1\|_{H\to H}\left\|\tilde{J}_1\right\|_{H\to H}\|R^{N-2}\|_{H\to H}\left\|\tilde{R}^{N-1}\right\|_{H\to H}\leq q$$

basit eşitsizliği elde edilir. Burada $q=|\alpha|+\dfrac{173}{66}|\beta|+\dfrac{173}{66}|\alpha||\beta|$ dır. $q<1$ hali için $I-B_n^\tau$ operatörünün sınırlı bir tersi

$$\left\|\left(I-B_n^{\tau}\right)^{-1}\right\|_{H\to H}\leq\frac{1}{1-q}=\frac{1}{1-|\alpha|-\frac{173}{66}|\beta|-\frac{173}{66}|\alpha||\beta|}$$

mevcuttur. Bu, Lemma 3.4 ü ispat eder. Şimdi problem (3.35) in çözüm formülünü elde edelim. 2.1 başlığındaki adımlar uygulanarak problem (3.35) in çözümü için

$$u_0=\mu,u_1=\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)^{-1}$$

$$\times\left(\left(I-\frac{5}{12}\tau^2A+\frac{\tau^4}{144}A^2\right)\mu+\tau\left(I-\frac{\tau^2}{12}A\right)\omega+\tau^2f_{2,2}\right),$$

$$u_k=\frac{1}{2}\left[R^k+\tilde{R}^k\right]\varphi+\frac{1}{2}\left[\tilde{R}^k-R^k\right]iA^{-1/2}\psi$$

$$+\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\left(\widehat{R}-R\right)^{-1}\left[\tilde{R}^k-R^k\right]\tau^2f_{2,2}$$

$$+\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\left(\widehat{R}-R\right)^{-1}\sum_{s=1}^{k-1}\left[\tilde{R}^{k-s}-R^{k-s}\right]\tau^2f_s \tag{3.41}$$

formülü elde edilir. Formül (3.41) den

$$\frac{u_k+u_{k-1}}{2}=\frac{1}{2}\left[J_1R^{k-1}+\widetilde{J}_1\tilde{R}^{k-1}\right]\mu+\frac{1}{2}\left[\widetilde{J}_1\tilde{R}^{k-1}-J_1R^{k-1}\right]iA^{-1/2}\omega$$

$$+\frac{1}{2}\left[\widetilde{J}_2\tilde{R}^{k-1}-J_2R^{k-1}\right]\tau^2f_{2,2}+\frac{1}{2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2f_{k-1}$$

$$+\frac{1}{2}\sum_{s=1}^{k-2}\left[\widehat{J}_2\tilde{R}^{k-1-s}-J_2R^{k-1-s}\right]\tau^2f_s \tag{3.42}$$

elde edilir. Lokal olmayan sınır şartları

$$\begin{cases} u_0 = \alpha u_N + \varphi, \\ \omega = \beta(I + i\tau A^{1/2})^{-1} \\ \times\left(\dfrac{85u_N - 108u_{N-1} + 27u_{N-2} - 4u_{N-3}}{66\tau} + \frac{3\tau}{11}(f_N - Au_N)\right) \\ +(I + i\tau A^{1/2})^{-1}\psi, \end{cases}$$

kullanılarak

$$\mu = \alpha\Big\{\frac{1}{2}\left[J_1 R^{N-1} + \widetilde{J}_1\tilde{R}^{N-1}\right]\mu + \frac{1}{2}\left[\widetilde{J}_1\tilde{R}^{N-1} - J_1 R^{N-1}\right] iA^{-1/2}\omega$$

$$+ \frac{1}{2}\left[\widetilde{J}_2\tilde{R}^{N-1} - J_2 R^{N-1}\right]\tau^2 f_{2,2} + \frac{1}{2}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1}\tau^2 f_{N-1}$$

$$+\frac{1}{2}\sum_{s=1}^{N-2}\left[\widehat{J}_2\tilde{R}^{N-1-s} - J_2 R^{N-1-s}\right]\tau^2 f_s\Big\} + \varphi \tag{3.43}$$

ve

$$\omega = \beta\left(I + i\tau A^{1/2}\right)^{-1}\left\{\frac{1}{2}\left(\left(-\frac{3\tau A}{11}R - J_7\right)J_1 R^{N-2} + \left(-\frac{3\tau A}{11}\widehat{R} - \widehat{J}_7\right)\widehat{J}_1\tilde{R}^{N-2}\right)\mu\right.$$

$$+ \frac{1}{2}\left(\left(-\frac{3\tau A}{11}\widetilde{R} - \widetilde{J}_7\right)\widetilde{J}_1\tilde{R}^{N-2} - \left(-\frac{3\tau A}{11}R - J_7\right)J_1 R^{N-2}\right) iA^{-1/2}\omega$$

$$+ \frac{1}{2}\left(\left(\frac{85}{66\tau} - \frac{3\tau A}{11}\right)f_{N-1} - \frac{108}{66\tau}f_{N-2} + \right.$$

$$\left.\frac{27}{66\tau}f_{N-3} - \frac{4}{66\tau}f_{N-4}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1}\tau^2$$

$$+\frac{1}{2}\Big(\Big(-\frac{3\tau A}{11}\tilde{R}-\tilde{J}_7\Big)\tilde{J}_2\tilde{R}^{N-2}-\Big(-\frac{3\tau A}{11}R-J_7\Big)J_2R^{N-2}\Big)\tau^2 f_{2,2}$$

$$+\frac{1}{2}\Big(\frac{85}{66\tau}-\frac{3\tau A}{11}\Big)\big(\tilde{J}_2\tilde{R}-J_2R\big)f_{N-2}\tau^2$$

$$+\frac{1}{2}\Big(\Big(\frac{85}{66\tau}-\frac{3\tau A}{11}\Big)\big(\tilde{J}_2\tilde{R}^2-J_2R^2\big)-\frac{108}{66\tau}\big(\tilde{J}_2\tilde{R}-J_2R\big)\Big)f_{N-3}\tau^2$$

$$+\frac{1}{2}\Big(\Big(\frac{85}{66\tau}-\frac{3\tau A}{11}\Big)\big(\tilde{J}_2\tilde{R}^3-J_2R^3\big)$$

$$-\frac{108}{66\tau}\big(\tilde{J}_2\tilde{R}^2-J_2R^2\big)+\frac{27}{66\tau}\big(\tilde{J}_2\tilde{R}-J_2R\big)\Big)f_{N-4}\tau^2$$

$$+\frac{1}{2}\sum_{s=1}^{N-5}\Big(\Big(-\frac{3\tau A}{11}\tilde{R}-\tilde{J}_7\Big)\tilde{J}_2\tilde{R}^{N-2-s}-\Big(-\frac{3\tau A}{11}R-J_7\Big)J_2R^{N-2-s}\Big)$$

$$\times\tau^2 f_s+\frac{3\tau}{11}f_N\Big\}+\big(I+i\tau A^{1/2}\big)^{-1}\psi \tag{3.44}$$

yazıldı. Formüller (3.43) ve (3.44) kullanılarak

$$\mu=T_\tau\Big\{\Big[\alpha\Big(\frac{1}{2}\big(\hat{J}_2\tilde{R}^{N-1}-J_2R^{N-1}\big)\tau^2 f_{2,2}+\frac{1}{2}\Big(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\Big)^{-1}\tau^2 f_{N-1}$$

$$+\frac{1}{2}\sum_{s=1}^{N-2}\big(\hat{J}_2\tilde{R}^{N-1-s}-J_2R^{N-1-s}\big)\tau^2 f_s\Big)+\varphi\Big]\Big[I-\frac{1}{2}\beta\big(I+i\tau A^{1/2}\big)^{-1}$$

$$\times\Big(\Big(-\frac{3i\tau A^{1/2}}{11}\tilde{R}+iA^{-1/2}\tilde{J}_7\Big)\tilde{J}_1\tilde{R}^{N-2}-\Big(-\frac{3i\tau A^{1/2}}{11}R+iA^{-1/2}J_7\Big)J_1R^{N-2}\Big)\Big]$$

$$+\left[\alpha\frac{1}{2}\left(\widetilde{J}_1\tilde{R}^{N-1}-J_1R^{N-1}\right)iA^{-1/2}\right]$$

$$\times\left[\beta(I+i\tau A^{1/2})^{-1}\left(\frac{1}{2}\left(\left(\frac{85}{66\tau}-\frac{3\tau A}{11}\right)f_{N-1}-\frac{108}{66\tau}f_{N-2}\right.\right.\right.$$

$$\left.+\frac{27}{66\tau}f_{N-3}-\frac{4}{66\tau}f_{N-4}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2$$

$$+\frac{1}{2}\left(\left(-\frac{3\tau A}{11}\widetilde{R}-\widetilde{J}_7\right)\widetilde{J}_2\tilde{R}^{N-2}-\left(-\frac{3\tau A}{11}R-J_7\right)J_2R^{N-2}\right)\tau^2f_{2,2}$$

$$+\frac{1}{2}\left(\frac{85}{66\tau}-\frac{3\tau A}{11}\right)\left(\widetilde{J}_2\tilde{R}-J_2R\right)f_{N-2}\tau^2$$

$$+\frac{1}{2}\left(\left(\frac{85}{66\tau}-\frac{3\tau A}{11}\right)\left(\widetilde{J}_2\tilde{R}^2-J_2R^2\right)-\frac{108}{66\tau}\left(\widetilde{J}_2\tilde{R}-J_2R\right)\right)f_{N-3}\tau^2$$

$$+\frac{1}{2}\left(\left(\frac{85}{66\tau}-\frac{3\tau A}{11}\right)\left(\widetilde{J}_2\tilde{R}^3-J_2R^3\right)\right.$$

$$\left.-\frac{108}{66\tau}\left(\widetilde{J}_2\tilde{R}^2-J_2R^2\right)+\frac{27}{66\tau}\left(\widetilde{J}_2\tilde{R}-J_2R\right)\right)f_{N-4}\tau^2$$

$$+\frac{1}{2}\sum_{s=1}^{N-5}\left(\left(-\frac{3\tau A}{11}\widetilde{R}-\widetilde{J}_7\right)\widetilde{J}_2\tilde{R}^{N-2-s}-\left(-\frac{3\tau A}{11}R-J_7\right)J_2R^{N-2-s}\right)\tau^2f_s$$

$$\left.\left.+\frac{3\tau}{11}f_N\right)+\left(I+i\tau A^{1/2}\right)^{-1}\psi\right]\Bigg\}\tag{3.45}$$

ve

$$\omega = T_\tau\Big\{\Big[I - \alpha\frac{1}{2}\big(J_1R^{N-1} + \tilde{J}_1\tilde{R}^{N-1}\big)\Big]$$

$$\times\Big[\beta(I + i\tau A^{1/2})^{-1}\Big(\frac{1}{2}\Big(\Big(\frac{85}{66\tau} - \frac{3\tau A}{11}\Big)f_{N-1} - \frac{108}{66\tau}f_{N-2}$$

$$+\frac{27}{66\tau}f_{N-3} - \frac{4}{66\tau}f_{N-4}\Big)\Big(I + \frac{\tau^2A}{12} + \frac{\tau^4A^2}{144}\Big)^{-1}\tau^2$$

$$+\frac{1}{2}\Big(\Big(-\frac{3\tau A}{11}\tilde{R} - \tilde{J}_7\Big)\tilde{J}_2\tilde{R}^{N-2} - \Big(-\frac{3\tau A}{11}R - J_7\Big)J_2R^{N-2}\Big)\tau^2f_{2,2}$$

$$+\frac{1}{2}\Big(\frac{85}{66\tau} - \frac{3\tau A}{11}\Big)\big(\tilde{J}_2\tilde{R} - J_2R\big)f_{N-2}\tau^2$$

$$+\frac{1}{2}\Big(\Big(\frac{85}{66\tau} - \frac{3\tau A}{11}\Big)\big(\tilde{J}_2\tilde{R}^2 - J_2R^2\big) - \frac{108}{66\tau}\big(\tilde{J}_2\tilde{R} - J_2R\big)\Big)$$

$$\times f_{N-3}\tau^2 + \frac{1}{2}\Big(\Big(\frac{85}{66\tau} - \frac{3\tau A}{11}\Big)\big(\tilde{J}_2\tilde{R}^3 - J_2R^3\big)$$

$$-\frac{108}{66\tau}\big(\tilde{J}_2\tilde{R}^2 - J_2R^2\big) + \frac{27}{66\tau}\big(\tilde{J}_2\tilde{R} - J_2R\big)\Big)f_{N-4}\tau^2$$

$$+\frac{1}{2}\sum_{s=1}^{N-5}\Big(\Big(-\frac{3\tau A}{11}\tilde{R} - \tilde{J}_7\Big)\tilde{J}_2\tilde{R}^{N-2-s} - \Big(-\frac{3\tau A}{11}R - J_7\Big)J_2R^{N-2-s}\Big)$$

$$\times\tau^2f_s + \frac{3\tau}{11}f_N\Big) + (I + i\tau A^{1/2})^{-1}\psi\Big]$$

$$+\left[\frac{1}{2}\beta\left(I + i\tau A^{1/2}\right)^{-1}\left(\left(-\frac{3\tau A}{11}R - J_7\right)J_1R^{N-2} + \left(-\frac{3\tau A}{11}\widehat{R} - \widehat{J}_7\right)\widehat{J}_1\tilde{R}^{N-2}\right)\right]$$

$$\times\left[\alpha\left(\frac{1}{2}\left(\widehat{J}_2\tilde{R}^{N-1}-J_2R^{N-1}\right)\tau^2 f_{2,2}+\frac{1}{2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2 f_{N-1}\right.\right.$$

$$\left.\left.+\frac{1}{2}\sum_{s=1}^{N-2}\left(\widehat{J}_2\tilde{R}^{N-1-s}-J_2R^{N-1-s}\right)\tau^2 f_s\right)+\varphi\right]\Bigg\} \tag{3.46}$$

elde edilir. (3.42), (3.45) ve (3.46) formülleri, problem (3.35) in bir çözümünü vermektedir.

Teorem 3.4 $\varphi\in D(A^{1/2}),\ \psi\in D(A^{\frac{1}{2}})$ olsun ve (3.36) koşulu ile Teorem 2.5 te verilen kararlılık kestirimleri sağlansın. Bu takdirde, fark şeması (3.35) in çözümü için aşağıdaki kararlılık kestirimleri sağlanır;

$$\max_{1\le k\le N}\left\|\frac{u_k+u_{k-1}}{2}\right\|_H\le M\left\{\sum_{s=1}^{k-1}\|A^{-1/2}f_s\|_H\tau+\|\varphi\|_H\right.$$

$$\left.+\left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{2,2}\right\|_H\right\}, \tag{3.47}$$

$$\max_{1\le k\le N}\left\|A^{1/2}\frac{u_k+u_{k-1}}{2}\right\|_H\le M\left\{\sum_{s=1}^{k-1}\|f_s\|_H\tau+\left\|A^{1/2}\varphi\right\|_H+\|\psi\|_H+\tau\left\|f_{2,2}\right\|_H\right\}, \tag{3.48}$$

$$\max_{1\le k\le N}\left\|A\frac{u_k+u_{k-1}}{2}\right\|_H\le M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H]+\|f_1\|_H+\|A\varphi\|_H\right.$$

$$\left.+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{2,2}\right\|_H\right\} \tag{3.49}$$

Burada M sabiti, $\tau,\ \varphi,\ \psi$, $f_{2,2}$ ve f_s $(1\le s\le N-1)$ ifadelerinden bağımsızdır.

İspat: (3.45), (3.46) formülleri, (3.36) koşulu, (2.73), (3.39) kestirimleri ve (3.40) eşitsizliği kullanılarak

$$\|\mu\|_H \le \|T_\tau\|_{H\to H}\Big\{\Big[|\alpha|\Big(\frac{1}{2}\Big(\|\tau A^{1/2}\tilde{J}_2\|_{H\to H}\|\tilde{R}^{N-1}\|_{H\to H}$$

$$+\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{N-1}\right\|_{H\to H}\Big)\tau\left\|A^{-1/2}f_{2,2}\right\|_H+\frac{1}{2}\left\|\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{-1/2}f_{N-1}\right\|_H$$

$$+\frac{1}{2}\sum_{s=1}^{N-2}\Big(\|\tau A^{1/2}\tilde{J}_2\|_{H\to H}\|\tilde{R}^{N-1-s}\|_{H\to H}+\|\tau A^{1/2}J_2\|_{H\to H}\|R^{N-1-s}\|_{H\to H}\Big)$$

$$\times\tau\left\|A^{-1/2}f_s\right\|_H\Big)+\|\varphi\|_H\Big]\Big[1+\frac{1}{2}|\beta|\Big(\Big(\frac{3}{11}\left\|i\tau A^{1/2}\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|\widehat{R}\right\|_{H\to H}$$

$$+\|(I+i\tau A^{1/2})^{-1}\|_{H\to H}\|iA^{-1/2}\tilde{J}_7\|_{H\to H}\Big)\|\tilde{J}_1\|_{H\to H}\|\tilde{R}^{N-2}\|_{H\to H}$$

$$+\Big(\frac{3}{11}\|i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\|_{H\to H}\|R\|_{H\to H}$$

$$+\|(I+i\tau A^{1/2})^{-1}\|_{H\to H}\|iA^{-1/2}J_7\|_{H\to H}\Big)\|J_1\|_{H\to H}\|R^{N-2}\|_{H\to H}\Big)\Big]$$

$$+\Big[\frac{1}{2}|\alpha|\Big(\|\tilde{J}_1\|_{H\to H}\|\tilde{R}^{N-1}\|_{H\to H}+\|J_1\|_{H\to H}\|R^{N-1}\|_{H\to H}\Big)|i|\Big]$$

$$\times\Bigg[|\beta|\Bigg(\frac{85}{132}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{-1/2}f_{N-1}\right\|_H$$

$$+\frac{9}{11}\|(I+i\tau A^{1/2})^{-1}\|_{H\to H}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|A^{-1/2}f_{N-2}\|_H$$

$$+\frac{1}{33}\|(I+i\tau A^{1/2})^{-1}\|_{H\to H}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|A^{-1/2}f_{N-4}\|_H$$

$$+\frac{1}{2}\Big(\Big(\frac{3}{11}\left\| i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|\widetilde{R}\|_{H\to H}$$

$$+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|iA^{-1/2}\widetilde{J}_7\|_{H\to H}\Big)\|\tau A^{1/2}\widetilde{J}_2\|_{H\to H}\|\widetilde{R}^{N-2}\|_{H\to H}$$

$$+\Big(\frac{3}{11}\left\| i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|R\|_{H\to H}$$

$$+\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|iA^{-1/2}J_7\right\|_{H\to H}\Big)\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{N-2}\right\|_{H\to H}\Big)\tau\left\|A^{-1/2}f_{2,2}\right\|_H$$

$$+\frac{85}{44}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}$$

$$\times\left\|\left(I-\frac{\tau^2A}{4}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}$$

$$\times\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\tau\|A^{-1/2}f_{N-2}\|_H$$

$$+\Big(\frac{85}{132}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}$$

$$\times\Big\|\Big(\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^5-\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^5\Big)$$

$$\times(i\tau A^{1/2})^{-1}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-2}\Big\|_{H\to H}$$

$$+\frac{27}{11}\left\|\left(I-\frac{\tau^2A}{4}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-2}\right\|_{H\to H}\Big)$$

$$\times \left\| (I + i\tau A^{1/2})^{-1} \right\|_{H\to H} \| A^{-1/2} f_{N-3} \|_{H\to H} \tau$$

$$+ \left(\frac{85}{132} \left\| \left(I - \frac{18}{85} \tau^2 A \right) \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-1} \right\|_{H\to H} \right.$$

$$\times \left\| \left(\left(I - \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^7 - \left(I + \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^7 \right) \right.$$

$$\left. \times \left(i\tau A^{1/2} \right)^{-1} \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-3} \right\|_{H\to H}$$

$$+ \frac{9}{11} \left\| \left(\left(I - \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^5 - \left(I + \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^5 \right) \right.$$

$$\left. \times \left(i\tau A^{1/2} \right)^{-1} \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-3} \right\|_{H\to H}$$

$$\left. + \frac{27}{44} \left\| \left(I - \frac{\tau^2 A}{4} + \frac{\tau^4 A^2}{144} \right) \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-1} \right\|_{H\to H} \right)$$

$$\times \left\| \left(I + i\tau A^{1/2} \right)^{-1} \right\|_{H\to H} \tau \| f_{N-4} \|_H + \frac{1}{2} \sum_{s=1}^{N-5} \left(\left(\frac{3}{11} \left\| i\tau A^{1/2} \left(I + i\tau A^{1/2} \right)^{-1} \right\|_{H\to H} \left\| \widehat{R} \right\|_{H\to H} \right. \right.$$

$$\left. + \left\| \left(I + i\tau A^{1/2} \right)^{-1} \right\|_{H\to H} \left\| i A^{-1/2} \widehat{J}_7 \right\|_{H\to H} \right) \left\| \tau A^{1/2} \widehat{J}_2 \right\|_{H\to H} \left\| \tilde{R}^{N-2-s} \right\|_{H\to H}$$

$$+ \left(\frac{3}{11} \left\| i\tau A^{1/2} (I + i\tau A^{1/2})^{-1} \right\|_{H\to H} \| R \|_{H\to H} \right.$$

$$+\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|iA^{-1/2}J_7\right\|_{H\to H}\Big)\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{N-2-s}\right\|_{H\to H}\Big)\tau\left\|A^{-1/2}f_s\right\|_H$$

$$+\frac{3}{11}\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{-1/2}f_N\right\|_H\Big)+\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|A^{-1/2}\psi\right\|_H\Big]\Big\}$$

$$\leq M\left\{\sum_{s=1}^{k-1}\left\|A^{-1/2}f_s\right\|_H\tau+\left\|\varphi\right\|_H+\left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{2,2}\right\|_H\right\} \tag{3.50}$$

ve

$$\left\|A^{-\frac{1}{2}}\omega\right\|_H\leq\left\|T_\tau\right\|_{H\to H}\left\{\left[1+\frac{1}{2}|\alpha|\left(\left\|J_1\right\|_{H\to H}\left\|R^{N-1}\right\|_{H\to H}+\left\|\widehat{J}_1\right\|_{H\to H}\left\|\tilde{R}^{N-1}\right\|_{H\to H}\right)\right]\right.$$

$$\times\left[|\beta|\left(\frac{85}{132}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{-1/2}f_{N-1}\right\|_H\right.\right.$$

$$+\frac{27}{33}\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{-1/2}f_{N-2}\right\|_H$$

$$+\frac{1}{33}\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{-1/2}f_{N-4}\right\|_H$$

$$+\frac{1}{2}\Big(\Big(\frac{3}{11}\left\|i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|\widetilde{R}\right\|_{H\to H}$$

$$+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|iA^{-1/2}\widetilde{J}_7\right\|_{H\to H}\Big)\left\|\tau A^{1/2}\widetilde{J}_2\right\|_{H\to H}\left\|\tilde{R}^{N-2}\right\|_{H\to H}$$

$$+\Big(\frac{3}{11}\left\|i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|R\right\|_{H\to H}$$

$$+\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|iA^{-1/2}J_7\right\|_{H\to H}\Big)\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{N-2}\right\|_{H\to H}\Big)\tau\left\|A^{-1/2}f_{2,2}\right\|_H$$

$$+\frac{85}{44}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}$$

$$\times\left\|\left(I-\frac{\tau^2A}{4}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}$$

$$\times\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\tau\|A^{-1/2}f_{N-2}\|_H$$

$$+\left(\frac{85}{132}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\right.$$

$$\times\left\|\left(\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^5-\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^5\right)\right.$$

$$\left.\times\left(i\tau A^{1/2}\right)^{-1}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-2}\right\|_{H\to H}$$

$$\left.+\frac{27}{11}\left\|\left(I-\frac{\tau^2A}{4}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-2}\right\|_{H\to H}\right)$$

$$\times\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|A^{-1/2}f_{N-3}\|_{H\to H}\tau$$

$$+\left(\frac{85}{132}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\right.$$

$$\times \left\| \left(\left(I - \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^7 - \left(I + \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^7 \right) \right.$$

$$\left. \times \left(i\tau A^{1/2} \right)^{-1} \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-3} \right\|_{H \to H}$$

$$+ \frac{9}{11} \left\| \left(\left(I - \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^5 - \left(I + \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^5 \right) \right.$$

$$\left. \times \left(i\tau A^{1/2} \right)^{-1} \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-3} \right\|_{H \to H}$$

$$\left. + \frac{27}{44} \left\| \left(I - \frac{\tau^2 A}{4} + \frac{\tau^4 A^2}{144} \right) \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-1} \right\|_{H \to H} \right)$$

$$\times \left\| \left(I + i\tau A^{1/2} \right)^{-1} \right\|_{H \to H} \tau \left\| f_{N-4} \right\|_H + \frac{1}{2} \sum_{s=1}^{N-5} \left(\left(\frac{3}{11} \left\| i\tau A^{1/2} \left(I + i\tau A^{1/2} \right)^{-1} \right\|_{H \to H} \left\| \widehat{R} \right\|_{H \to H} \right.\right.$$

$$\left. + \left\| (I + i\tau A^{1/2})^{-1} \right\|_{H \to H} \left\| iA^{-1/2} \tilde{J}_7 \right\|_{H \to H} \right) \left\| \tau A^{1/2} \tilde{J}_2 \right\|_{H \to H} \left\| \tilde{R}^{N-2-s} \right\|_{H \to H}$$

$$+ \left(\frac{3}{11} \left\| i\tau A^{1/2} (I + i\tau A^{1/2})^{-1} \right\|_{H \to H} \left\| R \right\|_{H \to H} \right.$$

$$\left.\left. + \left\| \left(I + i\tau A^{1/2} \right)^{-1} \right\|_{H \to H} \left\| iA^{-1/2} J_7 \right\|_{H \to H} \right) \left\| \tau A^{1/2} J_2 \right\|_{H \to H} \left\| R^{N-2-s} \right\|_{H \to H} \right) \tau \left\| A^{-1/2} f_s \right\|_H$$

$$\left. + \frac{3}{11} \left\| \left(I + i\tau A^{1/2} \right)^{-1} \right\|_{H \to H} \tau \left\| A^{-1/2} f_N \right\|_H \right) + \left\| \left(I + i\tau A^{1/2} \right)^{-1} \right\|_{H \to H} \left\| A^{-1/2} \psi \right\|_H \Bigg]$$

$$+\left[\frac{1}{2}|\beta|\Big(\Big(\frac{3}{11}\left\|\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|R\|_{H\to H}$$

$$+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|A^{-1/2}J_7\|_{H\to H}\Big)\|J_1\|_{H\to H}\|R^{N-2}\|_{H\to H}$$

$$+\Big(\frac{3}{11}\left\|\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|\tilde{R}\|_{H\to H}$$

$$+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|A^{-1/2}\tilde{J}_7\|_{H\to H}\Big)\|\tilde{J}_1\|_{H\to H}\|\tilde{R}^{N-2}\|_{H\to H}\Big)\right]$$

$$\times\Bigg[|\alpha|\Bigg(\frac{1}{2}\Big(\left\|\tau A^{1/2}\widehat{J}_2\right\|_{H\to H}\left\|\tilde{R}^{N-1}\right\|_{H\to H}+\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{N-1}\right\|_{H\to H}\Big)\tau\left\|A^{-1/2}f_{2,2}\right\|_H$$

$$+\frac{1}{2}\left\|\tau A^{1/2}\Big(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\Big)^{-1}\right\|_{H\to H}\tau\|A^{-1/2}f_{N-1}\|_H$$

$$+\frac{1}{2}\sum_{s=1}^{N-2}\Big(\|\tau A^{1/2}\tilde{J}_2\|_{H\to H}\|\tilde{R}^{N-1-s}\|_{H\to H}$$

$$+\|\tau A^{1/2}J_2\|_{H\to H}\|R^{N-1-s}\|_{H\to H}\Big)\tau\|A^{-1/2}f_s\|_H\Bigg)+\|\varphi\|_H\Bigg]\Bigg\}$$

$$\leq M\left\{\sum_{s=1}^{k-1}\left\|A^{-1/2}f_s\right\|_H\tau+\|\varphi\|_H+\left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{2,2}\right\|_H\right\} \qquad (3.51)$$

kararlılık kestirimleri elde edilir. Aynı metotla, (3.45) ve (3.46) formüllerine sırasıyla $A^{\frac{1}{2}}$ operatörü uygulanarak

$$\|A^{1/2}\mu\|_H\leq\|T_\tau\|_{H\to H}$$

$$\Big\{\Big[|\alpha|\Big(\frac{1}{2}\Big(\|\tau A^{1/2}\widetilde{J}_2\|_{H\to H}\|\tilde{R}^{N-1}\|_{H\to H}+\|\tau A^{1/2}J_2\|_{H\to H}\|R^{N-1}\|_{H\to H}\Big)\tau\|f_{2,2}\|_H$$

$$+\frac{1}{2}\left\|\tau A^{1/2}\Big(I+\frac{\tau^2 A}{12}+\frac{\tau^4A^2}{144}\Big)^{-1}\right\|_{H\to H}\tau\|f_{N-1}\|_H$$

$$+\frac{1}{2}\sum_{s=1}^{N-2}\Big(\left\|\tau A^{1/2}\widehat{J}_2\right\|_{H\to H}\left\|\tilde{R}^{N-1-s}\right\|_{H\to H}+\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{N-1-s}\right\|_{H\to H}\Big)\tau\left\|f_s\right\|_H\Big)+\left\|A^{1/2}\varphi\right\|_H\Big]$$

$$\times\Big[1+\frac{1}{2}|\beta|\Big(\Big(\frac{3}{11}\left\|i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|\widetilde{R}\|_{H\to H}$$

$$+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|iA^{-1/2}\widetilde{J}_7\|_{H\to H}\Big)\|\widetilde{J}_1\|_{H\to H}\|\tilde{R}^{N-2}\|_{H\to H}$$

$$+\Big(\frac{3}{11}\left\|i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|R\|_{H\to H}$$

$$+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|iA^{-1/2}J_7\|_{H\to H}\Big)\|J_1\|_{H\to H}\|R^{N-2}\|_{H\to H}\Big)\Big]$$

$$+\Big[\frac{1}{2}|\alpha|\Big(\|\widetilde{J}_1\|_{H\to H}\|\tilde{R}^{N-1}\|_{H\to H}+\|J_1\|_{H\to H}\|R^{N-1}\|_{H\to H}\Big)|i|\Big]$$

$$\times\left[|\beta|\left(\frac{85}{132}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\tau\left\|f_{N-1}\right\|_H\right.\right.$$

$$+\frac{27}{33}\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|\Big(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\Big)^{-1}\right\|_{H\to H}\tau\|f_{N-2}\|_H$$

$$+\frac{1}{33}\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|f_{N-4}\|_H$$

$$+\frac{1}{2}\left(\left(\frac{3}{11}\left\|i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|\widetilde{R}\|_{H\to H}\right.\right.$$

$$\left.+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|iA^{-1/2}\widetilde{J}_7\|_{H\to H}\right)\|\tau A^{1/2}\widetilde{J}_2\|_{H\to H}\|\widetilde{R}^{N-2}\|_{H\to H}$$

$$+\left(\frac{3}{11}\left\|i\tau A^{1/2}\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\|R\|_{H\to H}\right.$$

$$\left.\left.+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|iA^{-1/2}J_7\|_{H\to H}\right)\|\tau A^{1/2}J_2\|_{H\to H}\|R^{N-2}\|_{H\to H}\right)\tau\|f_{2,2}\|_H$$

$$+\frac{85}{44}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}$$

$$\times\left\|\left(I-\frac{\tau^2A}{4}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{-1/2}f_{N-2}\right\|_H$$

$$+\left(\frac{85}{132}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\right.$$

$$\times\left\|\left(\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^5-\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^5\right)\right.$$

$$\left.\times\left(i\tau A^{1/2}\right)^{-1}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-2}\right\|_{H\to H}$$

$$+\frac{27}{11}\left\|\left(I-\frac{\tau^2A}{4}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-2}\right\|_{H\to H}\Bigg)$$

$$\times\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|A^{-1/2}f_{N-3}\|_{H\to H}\tau$$

$$+\left(\frac{85}{132}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\right.$$

$$\times\left\|\left(\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^7-\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^7\right)\right.$$

$$\left.\times\left(i\tau A^{1/2}\right)^{-1}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-3}\right\|_{H\to H}$$

$$+\frac{9}{11}\left\|\left(\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^5-\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^5\right)\right.$$

$$\left.\times\left(i\tau A^{1/2}\right)^{-1}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-3}\right\|_{H\to H}$$

$$+\frac{27}{44}\left\|\left(I-\frac{\tau^2A}{4}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\Bigg)$$

$$\times\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\tau\|f_{N-4}\|_H+\frac{1}{2}\sum_{s=1}^{N-5}\left(\left(\frac{3}{11}\left\|i\tau A^{1/2}\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\|\widehat{R}\|_{H\to H}\right.\right.$$

$$\left.+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|iA^{-1/2}\widetilde{\mathfrak{I}}_7\right\|_{H\to H}\right)\left\|\tau A^{1/2}\widetilde{\mathfrak{I}}_2\right\|_{H\to H}\|\tilde{R}^{N-2-s}\|_{H\to H}$$

$$+\left(\frac{3}{11}\left\| i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|R\|_{H\to H}\right.$$

$$+\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|iA^{-1/2}J_7\right\|_{H\to H}\Big)\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{N-2-s}\right\|_{H\to H}\Big)\tau\left\|f_s\right\|_H$$

$$+\frac{3}{11}\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\tau\|f_N\|_H\Big)+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|\psi\|_H\Big]\Big\}$$

$$\leq M\left\{\sum_{s=1}^{k-1}\|f_s\|_H\,\tau+\left\|A^{1/2}\varphi\right\|_H+\|\psi\|_H+\tau\left\|f_{2,2}\right\|_H\right\} \tag{3.52}$$

ve

$$\|\omega\|_H\leq\|T_\tau\|_{H\to H}\left\{\left[1+\frac{1}{2}|\alpha|\left(\|J_1\|_{H\to H}\left\|R^{N-1}\right\|_{H\to H}+\left\|\widehat{J}_1\right\|_{H\to H}\left\|\tilde{R}^{N-1}\right\|_{H\to H}\right)\right]\right.$$

$$\times\left[|\beta|\left(\frac{85}{132}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\tau\left\|f_{N-1}\right\|_H\right.\right.$$

$$+\frac{27}{33}\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|f_{N-2}\|_H$$

$$+\frac{1}{33}\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|f_{N-4}\|_H$$

$$+\frac{1}{2}\left(\left(\frac{3}{11}\left\|i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|\widetilde{R}\|_{H\to H}\right.\right.$$

$$+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|iA^{-1/2}\widetilde{J}_7\right\|_{H\to H}\Big)\left\|\tau A^{1/2}\widetilde{J}_2\right\|_{H\to H}\|\tilde{R}^{N-2}\|_{H\to H}$$

$$+\left(\frac{3}{11}\left\| i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|R\|_{H\to H}\right.$$

$$\left.\left.+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|iA^{-1/2}J_7\|_{H\to H}\right)\|\tau A^{1/2}J_2\|_{H\to H}\|R^{N-2}\|_{H\to H}\right)\tau\|f_{2,2}\|_H$$

$$+\frac{85}{44}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}$$

$$\times\left\|\left(I-\frac{\tau^2A}{4}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{-1/2}f_{N-2}\right\|_H$$

$$+\left(\frac{85}{132}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\right.$$

$$\times\left\|\left(\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^5-\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^5\right)\right.$$

$$\left.\times\left(i\tau A^{1/2}\right)^{-1}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-2}\right\|_{H\to H}$$

$$\left.+\frac{27}{11}\left\|\left(I-\frac{\tau^2A}{4}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-2}\right\|_{H\to H}\right)$$

$$\times\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|A^{-1/2}f_{N-3}\right\|_{H\to H}\tau+\left(\frac{85}{132}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\right.$$

$$\times\left\|\left(\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^7-\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^7\right)\right.$$

$$\times\left(i\tau A^{1/2}\right)^{-1}\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-3}\Bigg\|_{H\to H}$$

$$+\ \frac{9}{11}\ \Bigg\|\left(\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2 A}{12}\right)^5-\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2 A}{12}\right)^5\right)$$

$$\times\left(i\tau A^{1/2}\right)^{-1}\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-3}\Bigg\|_{H\to H}$$

$$+\frac{27}{44}\left\|\left(I-\frac{\tau^2 A}{4}+\frac{\tau^4 A^2}{144}\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H}\Bigg)\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\tau\|f_{N-4}\|_H$$

$$+\ \frac{1}{2}\sum_{s=1}^{N-5}\Bigg(\Bigg(\frac{3}{11}\ \|\,i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\,\|_{H\to H}\|\widetilde{R}\,\|_{H\to H}$$

$$+\|\,(I+i\tau A^{1/2})^{-1}\,\|_{H\to H}\|\,iA^{-1/2}\widetilde{J}_7\,\|_{H\to H}\Bigg)\,\|\,\tau A^{1/2}\widetilde{J}_2\,\|_{H\to H}\|\widetilde{R}^{N-2-s}\,\|_{H\to H}$$

$$+\ \Bigg(\frac{3}{11}\ \|\,i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\,\|_{H\to H}\|R\|_{H\to H}$$

$$+\|\,(I+i\tau A^{1/2})^{-1}\,\|_{H\to H}\|iA^{-1/2}J_7\|_{H\to H}\Bigg)\|\tau A^{1/2}J_2\|_{H\to H}\|R^{N-2-s}\,\|_{H\to H}\Bigg)\tau\|f_s\|_H$$

$$+\frac{3}{11}\ \|\,(I+i\tau A^{1/2})^{-1}\,\|_{H\to H}\tau\|f_N\|_H\Bigg)+\ \|\,(I+i\tau A^{1/2})^{-1}\,\|_{H\to H}\|\psi\|_H\Bigg]$$

$$+\Bigg[\ \frac{1}{2}|\beta|\Bigg(\Bigg(\frac{3}{11}\ \|\,\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\,\|_{H\to H}\|R\|_{H\to H}$$

$$+\left\| (I+i\tau A^{1/2})^{-1} \right\|_{H\to H} \| A^{-1/2} J_7 \|_{H\to H} \Big) \| J_1 \|_{H\to H} \| R^{N-2} \|_{H\to H}$$

$$+\Big(\frac{3}{11} \left\| \tau A^{1/2} (I+i\tau A^{1/2})^{-1} \right\|_{H\to H} \| \widetilde{R} \|_{H\to H}$$

$$+\left\| (I+i\tau A^{1/2})^{-1} \right\|_{H\to H} \| A^{-1/2} \widetilde{J}_7 \|_{H\to H} \Big) \| \widetilde{J}_1 \|_{H\to H} \| \tilde{R}^{N-2} \|_{H\to H} \Big) \Big]$$

$$\times \Big[|\alpha| \Big(\frac{1}{2} \Big(\| \tau A^{1/2} \widetilde{J}_2 \|_{H\to H} \| \tilde{R}^{N-1} \|_{H\to H}$$

$$+\left\| \tau A^{1/2} J_2 \right\|_{H\to H} \left\| R^{N-1} \right\|_{H\to H} \Big) \tau \left\| f_{2,2} \right\|_H + \frac{1}{2} \left\| \tau A^{1/2} \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-1} \right\|_{H\to H} \tau \left\| f_{N-1} \right\|_H$$

$$+\frac{1}{2} \sum_{s=1}^{N-2} \Big(\| \tau A^{1/2} \widetilde{J}_2 \|_{H\to H} \| \tilde{R}^{N-1-s} \|_{H\to H}$$

$$+\| \tau A^{1/2} J_2 \|_{H\to H} \| R^{N-1-s} \|_{H\to H} \Big) \tau \| A^{-1/2} f_s \|_H \Big) + \| A^{1/2} \varphi \|_H \Big] \Big\}$$

$$\leq M \left\{ \sum_{s=1}^{k-1} \left\| f_s \right\|_H \tau + \left\| A^{1/2} \varphi \right\|_H + \left\| \psi \right\|_H + \tau \left\| f_{2,2} \right\|_H \right\} \tag{3.53}$$

kararlılık kestirimleri elde edilir. Daha sonra (3.45) ve (3.46) ya Abel formülü uygulanarak

$$\mu = T_\tau \Bigg\{ \Bigg[\alpha \left(\frac{1}{2} \left(\widehat{J}_2 \tilde{R}^{N-1} - J_2 R^{N-1} \right) \tau^2 f_{2,2} + \frac{1}{2} \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-1} \tau^2 f_{N-1} \right.$$

$$\left. + \frac{1}{2} A^{-1} \left(-\sum_{s=2}^{N-2} \left(\tilde{R}^{N-s} + R^{N-s} \right) \left(f_s - f_{s-1} \right) + 2 f_{N-2} - \left(\tilde{R}^{N-1} + R^{N-1} \right) f_1 \right) + \varphi \right]$$

$$\times\Big[I-\frac{1}{2}\beta(I+i\tau A^{1/2})^{-1}\Big(\Big(-\frac{3i\tau A^{1/2}}{11}\tilde{R}+iA^{-1/2}\tilde{J}_7\Big)\tilde{J}_1\tilde{R}^{N-2}$$

$$-\left(-\frac{3i\tau A^{1/2}}{11}R+iA^{-1/2}J_7\right)J_1R^{N-2}\right)\Big]+\left[\alpha\frac{1}{2}\left(\widehat{J}_1\tilde{R}^{N-1}-J_1R^{N-1}\right)iA^{-1/2}\right]$$

$$\times\Big[\beta(I+i\tau A^{1/2})^{-1}\Big(\frac{1}{2}\Big(\Big(\frac{85}{66\tau}-\frac{3\tau A}{11}\Big)f_{N-1}-\frac{108}{66\tau}f_{N-2}$$

$$+\frac{27}{66\tau}f_{N-3}-\frac{4}{66\tau}f_{N-4}\Big)\Big(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\Big)^{-1}\tau^2$$

$$+\frac{1}{2}\Big(\Big(-\frac{3\tau A}{11}\tilde{R}-\tilde{J}_7\Big)\tilde{J}_2\tilde{R}^{N-2}-\Big(-\frac{3\tau A}{11}R-J_7\Big)J_2R^{N-2}\Big)\tau^2f_{2,2}$$

$$+\frac{1}{2}\Big(\frac{85}{66\tau}-\frac{3\tau A}{11}\Big)\big(\tilde{J}_2\tilde{R}-J_2R\big)f_{N-2}\tau^2$$

$$+\frac{1}{2}\Big(\Big(\frac{85}{66\tau}-\frac{3\tau A}{11}\Big)\big(\tilde{J}_2\tilde{R}^2-J_2R^2\big)-\frac{108}{66\tau}\big(\tilde{J}_2\tilde{R}-J_2R\big)\Big)f_{N-3}\tau^2$$

$$+\frac{1}{2}\Big(\Big(\frac{85}{66\tau}-\frac{3\tau A}{11}\Big)\big(\tilde{J}_2\tilde{R}^3-J_2R^3\big)$$

$$-\frac{108}{66\tau}\big(\tilde{J}_2\tilde{R}^2-J_2R^2\big)+\frac{27}{66\tau}\big(\tilde{J}_2\tilde{R}-J_2R\big)\Big)f_{N-4}\tau^2$$

$$+\frac{1}{2}A^{-1}\Bigg(-\sum_{s=2}^{N-5}\Big(\Big(-\frac{3\tau A}{11}\tilde{R}-\tilde{J}_7\Big)\tilde{R}^{N-1-s}+\Big(-\frac{3\tau A}{11}R-J_7\Big)R^{N-1-s}\Big)$$

$$\times(f_s-f_{s-1})+\Big(\Big(-\frac{3\tau A}{11}\tilde{R}-\tilde{J}_7\Big)\tilde{R}^2+\Big(-\frac{3\tau A}{11}R-J_7\Big)R^2\Big)f_{N-5}$$

$$-\Big(\Big(-\frac{3\tau A}{11}\tilde{R}-\tilde{J}_7\Big)\tilde{R}^{N-2}+\Big(-\frac{3\tau A}{11}R-J_7\Big)R^{N-2}\Big)f_1\Big)$$

$$+\frac{3\tau}{11}f_N\Bigg)+\left(I+i\tau A^{1/2}\right)^{-1}\psi\Bigg]\Bigg\} \tag{3.54}$$

ve

$$\omega=T_\tau\Big\{\Big[I-\alpha\frac{1}{2}\Big(J_1R^{N-1}+\tilde{J}_1\tilde{R}^{N-1}\Big)\Big]$$

$$\times\Big[\beta(I+i\tau A^{1/2})^{-1}\Big(\frac{1}{2}\Big(\Big(\frac{85}{66\tau}-\frac{3\tau A}{11}\Big)f_{N-1}-\frac{108}{66\tau}f_{N-2}$$

$$+\frac{27}{66\tau}f_{N-3}-\frac{4}{66\tau}f_{N-4}\Big)\Big(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\Big)^{-1}\tau^2$$

$$+\frac{1}{2}\Big(\Big(-\frac{3\tau A}{11}\tilde{R}-\tilde{J}_7\Big)\tilde{J}_2\tilde{R}^{N-2}-\Big(-\frac{3\tau A}{11}R-J_7\Big)J_2R^{N-2}\Big)\tau^2f_{2,2}$$

$$+\frac{1}{2}\Big(\frac{85}{66\tau}-\frac{3\tau A}{11}\Big)\Big(\tilde{J}_2\tilde{R}-J_2R\Big)f_{N-2}\tau^2$$

$$+\frac{1}{2}\Big(\Big(\frac{85}{66\tau}-\frac{3\tau A}{11}\Big)\Big(\tilde{J}_2\tilde{R}^2-J_2R^2\Big)-\frac{108}{66\tau}\Big(\tilde{J}_2\tilde{R}-J_2R\Big)\Big)f_{N-3}\tau^2$$

$$+\frac{1}{2}\Big(\Big(\frac{85}{66\tau}-\frac{3\tau A}{11}\Big)\Big(\tilde{J}_2\tilde{R}^3-J_2R^3\Big)$$

$$-\frac{108}{66\tau}\Big(\tilde{J}_2\tilde{R}^2-J_2R^2\Big)+\frac{27}{66\tau}\Big(\tilde{J}_2\tilde{R}-J_2R\Big)\Big)f_{N-4}\tau^2$$

$$+\frac{1}{2}A^{-1}\Bigg(-\sum_{s=2}^{N-5}\Big(\Big(-\frac{3\tau A}{11}\widetilde{R}-\widetilde{J}_7\Big)\tilde{R}^{N-1-s}+\Big(-\frac{3\tau A}{11}R-J_7\Big)R^{N-1-s}\Big)$$

$$\times(f_s-f_{s-1})+\Big(\Big(-\frac{3\tau A}{11}\widetilde{R}-\widetilde{J}_7\Big)\tilde{R}^2+\Big(-\frac{3\tau A}{11}R-J_7\Big)R^2\Big)f_{N-5}$$

$$-\left(\left(-\frac{3\tau A}{11}\widehat{R}-\widehat{J}_7\right)\tilde{R}^{N-2}+\left(-\frac{3\tau A}{11}R-J_7\right)R^{N-2}\right)f_1\Bigg)+\frac{3\tau}{11}f_N\Bigg)+\left(I+i\tau A^{1/2}\right)^{-1}\psi\Bigg]$$

$$+\Big[\frac{1}{2}\beta(I+i\tau A^{1/2})^{-1}\Big(\Big(-\frac{3\tau A}{11}R-J_7\Big)J_1R^{N-2}+\Big(-\frac{3\tau A}{11}\widetilde{R}-\widetilde{J}_7\Big)\widetilde{J}_1\tilde{R}^{N-2}\Big)\Big]$$

$$\times\Bigg[\alpha\left(\frac{1}{2}\left(\widehat{J}_2\tilde{R}^{N-1}-J_2R^{N-1}\right)\tau^2f_{2,2}+\frac{1}{2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2f_{N-1}\right.$$

$$\left.+\frac{1}{2}A^{-1}\left(-\sum_{s=2}^{N-2}\left(\tilde{R}^{N-s}+R^{N-s}\right)\left(f_s-f_{s-1}\right)+2f_{N-2}-\left(\tilde{R}^{N-1}+R^{N-1}\right)f_1\right)+\varphi\right]\Bigg\} \tag{3.55}$$

elde edilir. Şimdi $\|A\mu\|_H$, $\|A^{1/2}\omega\|_H$ için kararlılık kestirimleri elde edelim.

Öncelikle, (3.54) formülüne, (2.73), (3.39) kestirimleri ve (3.40) eşitsizliğinden yararlanarak, A operatörü ve üçgen eşitsizliği uygulanırsa,

$$\|A\mu\|_H\le\|T_\tau\|_{H\to H}\Bigg\{\Bigg[|\alpha|\Bigg(\frac{1}{2}\Big(\left\|\tau A^{1/2}\widehat{J}_2\right\|_{H\to H}\left\|\tilde{R}^{N-1}\right\|_{H\to H}+\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{N-1}\right\|_{H\to H}\Big)$$

$$\times\tau\|A^{1/2}f_{2,2}\|_H+\frac{1}{2}\left\|\tau A^{1/2}\Big(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\Big)^{-1}\right\|_{H\to H}\tau\|A^{1/2}f_{N-1}\|_H$$

$$+\frac{1}{2}\Bigg(\sum_{s=2}^{N-2}\Big(\|\tilde{R}^{N-s}\|_{H\to H}+\|R^{N-s}\|_{H\to H}\Big)\|f_s-f_{s-1}\|_H+2\|f_{N-2}\|_H$$

$$+\left(\|\tilde{R}^{N-1}\|_{H\to H}+\|R^{N-1}\|_{H\to H}\right)\|f_1\|_H\Big)+\|A\varphi\|_H\Big]$$

$$\times\Big[1+\frac{1}{2}|\beta|\Big(\Big(\frac{3}{11}\left\|i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|\widetilde{R}\|_{H\to H}$$

$$+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|iA^{-1/2}\mathfrak{J}_7\|_{H\to H}\Big)\|\mathfrak{J}_1\|_{H\to H}\|\tilde{R}^{N-2}\|_{H\to H}$$

$$+\Big(\frac{3}{11}\left\|i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|R\|_{H\to H}$$

$$+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|iA^{-1/2}J_7\|_{H\to H}\Big)\|J_1\|_{H\to H}\|R^{N-2}\|_{H\to H}\Big)\Big]$$

$$+\Big[\frac{1}{2}|\alpha|\Big(\|\mathfrak{J}_1\|_{H\to H}\|\tilde{R}^{N-1}\|_{H\to H}+\|J_1\|_{H\to H}\|R^{N-1}\|_{H\to H}\Big)|i|\Big]$$

$$\times\left[|\beta|\left(\frac{85}{132}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{1/2}f_{N-1}\right\|_H\right.\right.$$

$$+\frac{9}{11}\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|A^{1/2}f_{N-2}\|_H$$

$$+\frac{1}{33}\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|A^{1/2}f_{N-4}\|_H$$

$$+\frac{1}{2}\Big(\Big(\frac{3}{11}\left\|i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|\widetilde{R}\|_{H\to H}$$

$$+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|iA^{-1/2}\mathfrak{J}_7\|_{H\to H}\Big)\|\tau A^{1/2}\mathfrak{J}_2\|_{H\to H}\|\tilde{R}^{N-2}\|_{H\to H}$$

$$+\left(\frac{3}{11}\left\| i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|R\|_{H\to H}\right.$$

$$+\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|iA^{-1/2}J_7\right\|_{H\to H}\Big)\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{N-2}\right\|_{H\to H}\Big)\tau\left\|A^{1/2}f_{2,2}\right\|_H$$

$$+\frac{85}{44}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}$$

$$\times\left\|\left(I-\frac{\tau^2A}{4}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{1/2}f_{N-2}\right\|_H$$

$$+\left(\frac{85}{132}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\right.$$

$$\times\left\|\left(\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^5-\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^5\right)\right.$$

$$\left.\times\left(i\tau A^{1/2}\right)^{-1}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-2}\right\|_{H\to H}$$

$$\left.+\frac{27}{11}\left\|\left(I-\frac{\tau^2A}{4}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-2}\right\|_{H\to H}\right)$$

$$\times\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|A^{1/2}f_{N-3}\|_{H\to H}\tau$$

$$+\left(\frac{85}{132}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\right.$$

$$\times \left\| \left(\left(I - \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^7 - \left(I + \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^7 \right) \right.$$

$$\left. \times \left(i\tau A^{1/2} \right)^{-1} \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-3} \right\|_{H \to H}$$

$$+ \frac{9}{11} \left\| \left(\left(I - \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^5 - \left(I + \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12} \right)^5 \right) \right.$$

$$\left. \times \left(i\tau A^{1/2} \right)^{-1} \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-3} \right\|_{H \to H}$$

$$+ \frac{27}{44} \left\| \left(I - \frac{\tau^2 A}{4} + \frac{\tau^4 A^2}{144} \right) \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-1} \right\|_{H \to H} \Bigg)$$

$$\times \left\| \left(I + i\tau A^{1/2} \right)^{-1} \right\|_{H \to H} \tau \left\| A^{1/2} f_{N-4} \right\|_H + \frac{1}{2} \left(\sum_{s=2}^{N-5} \left(\left(\frac{3}{11} \left\| i\tau A^{1/2} \left(I + i\tau A^{1/2} \right)^{-1} \right\|_{H \to H} \left\| \widehat{R} \right\|_{H \to H} \right. \right. \right.$$

$$\left. + \left\| (I + i\tau A^{1/2})^{-1} \right\|_{H \to H} \left\| iA^{-1/2} \tilde{J}_7 \right\|_{H \to H} \right) \left\| \tilde{R}^{N-1-s} \right\|_{H \to H}$$

$$+ \left(\frac{3}{11} \left\| i\tau A^{1/2} (I + i\tau A^{1/2})^{-1} \right\|_{H \to H} \left\| R \right\|_{H \to H} \right.$$

$$\left. \left. + \left\| (I + i\tau A^{1/2})^{-1} \right\|_{H \to H} \left\| iA^{-1/2} J_7 \right\|_{H \to H} \right) \left\| R^{N-1-s} \right\|_{H \to H} \right) \left\| f_s - f_{s-1} \right\|_H$$

$$+ \left(\left(\frac{3}{11} \left\| i\tau A^{1/2} \left(I + i\tau A^{1/2} \right)^{-1} \right\|_{H \to H} \left\| \widehat{R}^2 \right\|_{H \to H} + \left\| \left(I + i\tau A^{1/2} \right)^{-1} \right\|_{H \to H} \left\| iA^{-1/2} \widehat{J}_7 \right\|_{H \to H} \right) \right.$$

$$+\left(\frac{3}{11}\left\| i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|R^2\|_{H\to H}\right.$$

$$\left.\left.+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|iA^{-1/2}J_7\|_{H\to H}\right)\right)\|f_{N-5}\|_H$$

$$+\left(\left(\frac{3}{11}\left\| i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|\widetilde{R}^{N-2}\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|iA^{-1/2}\widetilde{J}_7\|_{H\to H}\right)$$

$$+\left(\frac{3}{11}\left\|i\tau A^{1/2}\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|R^{N-2}\right\|_{H\to H}+\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|iA^{-1/2}J_7\right\|_{H\to H}\right)\left\|f_1\right\|_H\right)$$

$$\left.\left.+\frac{3}{11}\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{1/2}f_N\right\|_H\right)+\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|A^{1/2}\psi\right\|_H\right]\right\}$$

$$\leq M\left\{\sum_{s=2}^{N-1}\left\|f_s-f_{s-1}\right\|_H+\left\|f_1\right\|_H+\left\|A\varphi\right\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{2,2}\right\|_H\right\} \qquad (3.56)$$

kararlılık kestirimi elde edilir. İkinci adımda formül (3.55) e $A^{1/2}$ operatörü ve üçgen eşitsizliği uygulanarak, (2.73), (3.39) kestirimleri ve (3.40) eşitsizliği yardımıyla

$$\left\|A^{1/2}\omega\right\|_H\leq\left\|T_\tau\right\|_{H\to H}\left\{\left[1+\frac{1}{2}|\alpha|\left(\left\|J_1\right\|_{H\to H}\left\|R^{N-1}\right\|_{H\to H}+\left\|\widehat{J}_1\right\|_{H\to H}\left\|\tilde{R}^{N-1}\right\|_{H\to H}\right)\right]\right.$$

$$\times\left[|\beta|\left(\frac{85}{132}\left\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{1/2}f_{N-1}\right\|_H\right.\right.$$

$$+\frac{27}{33}\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|A^{1/2}f_{N-2}\|_H$$

$$+\frac{1}{33}\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\left\|\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|A^{1/2}f_{N-4}\|_H$$

$$+\frac{1}{2}\left(\left(\frac{3}{11}\left\|i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|\widetilde{R}\|_{H\to H}\right.\right.$$

$$\left.+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|iA^{-1/2}\widetilde{J}_7\|_{H\to H}\right)\|\tau A^{1/2}\widetilde{J}_2\|_{H\to H}\|\tilde{R}^{N-2}\|_{H\to H}$$

$$+\left(\frac{3}{11}\left\|i\tau A^{1/2}(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|R\|_{H\to H}\right.$$

$$\left.\left.+\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|iA^{-1/2}J_7\right\|_{H\to H}\right)\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{N-2}\right\|_{H\to H}\right)\tau\left\|A^{1/2}f_{2,2}\right\|_H$$

$$+\frac{85}{44}\left\|\left(I-\frac{18}{85}\tau^2 A\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H}$$

$$\times\left\|\left(I-\frac{\tau^2 A}{4}+\frac{\tau^4 A^2}{144}\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{1/2}f_{N-2}\right\|_H$$

$$+\left(\frac{85}{132}\left\|\left(I-\frac{18}{85}\tau^2 A\right)\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H}\right.$$

$$\times\left\|\left(\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2 A}{12}\right)^5-\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2 A}{12}\right)^5\right)\right.$$

$$\times\left(i\tau A^{1/2}\right)^{-1}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-2}\Bigg\|_{H\to H}$$

$$+\frac{27}{11}\left\|\left(I-\frac{\tau^2A}{4}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-2}\right\|_{H\to H}\Bigg)\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|A^{1/2}f_{N-3}\right\|_{H\to H}\tau$$

$$+\Bigg(\frac{85}{132}\Bigg\|\left(I-\frac{18}{85}\tau^2A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\Bigg\|_{H\to H}$$

$$\times\Bigg\|\left(\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^7-\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^7\right)$$

$$\times\left(i\tau A^{1/2}\right)^{-1}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-3}\Bigg\|_{H\to H}$$

$$+\frac{9}{11}\Bigg\|\left(\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^5-\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^5\right)$$

$$\times\left(i\tau A^{1/2}\right)^{-1}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-3}\Bigg\|_{H\to H}$$

$$+\frac{27}{44}\left\|\left(I-\frac{\tau^2A}{4}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\Bigg)$$

$$\times\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{1/2}f_{N-4}\right\|_{H}+\frac{1}{2}\left(\sum_{s=2}^{N-5}\left(\left(\frac{3}{11}\left\|i\tau A^{1/2}\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|\widehat{R}\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|iA^{-1/2}\widehat{J}_7\right\|_{H\to H}\right)\left\|\tilde{R}^{N-1-s}\right\|_{H\to H}+\left(\frac{3}{11}\left\|i\tau A^{1/2}\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|R\right\|_{H\to H}\right.$$

$$+\left\| (I+i\tau A^{1/2})^{-1} \right\|_{H\to H} \| iA^{-1/2}J_7 \|_{H\to H} \Big) \| R^{N-1-s} \|_{H\to H} \Big) \| f_s - f_{s-1} \|_H$$

$$+\Big(\Big(\frac{3}{11} \left\| i\tau A^{1/2}(I+i\tau A^{1/2})^{-1} \right\|_{H\to H} \left\| \widetilde{R}^2 \right\|_{H\to H}$$

$$+\left\| \left(I+i\tau A^{1/2}\right)^{-1} \right\|_{H\to H} \left\| iA^{-1/2}\widehat{J}_7 \right\|_{H\to H} \Big) + \Big(\frac{3}{11} \left\| i\tau A^{1/2}\left(I+i\tau A^{1/2}\right)^{-1} \right\|_{H\to H} \left\| R^2 \right\|_{H\to H}$$

$$+\left\| (I+i\tau A^{1/2})^{-1} \right\|_{H\to H} \| iA^{-1/2}J_7 \|_{H\to H} \Big) \Big) \| f_{N-5} \|_H$$

$$+\Big(\Big(\frac{3}{11} \left\| i\tau A^{1/2}(I+i\tau A^{1/2})^{-1} \right\|_{H\to H} \left\| \widetilde{R}^{N-2} \right\|_{H\to H}$$

$$+\left\| \left(I+i\tau A^{1/2}\right)^{-1} \right\|_{H\to H} \left\| iA^{-1/2}\widehat{J}_7 \right\|_{H\to H} \Big) + \Big(\frac{3}{11} \left\| i\tau A^{1/2}\left(I+i\tau A^{1/2}\right)^{-1} \right\|_{H\to H} \left\| R^{N-2} \right\|_{H\to H}$$

$$+\left\| (I+i\tau A^{1/2})^{-1} \right\|_{H\to H} \| iA^{-1/2}J_7 \|_{H\to H} \Big) \Big) \| f_1 \|_H \Big)$$

$$+\frac{3}{11} \left\| \left(I+i\tau A^{1/2}\right)^{-1} \right\|_{H\to H} \tau \left\| A^{-1/2} f_N \right\|_H \Big) + \left\| \left(I+i\tau A^{1/2}\right)^{-1} \right\|_{H\to H} \left\| A^{1/2}\psi \right\|_H \Big]$$

$$+\Big[\frac{1}{2}|\beta| \Big(\Big(\frac{3}{11} \left\| \tau A^{1/2}(I+i\tau A^{1/2})^{-1} \right\|_{H\to H} \| R \|_{H\to H}$$

$$+\left\| (I+i\tau A^{1/2})^{-1} \right\|_{H\to H} \| A^{-1/2}J_7 \|_{H\to H} \Big) \| J_1 \|_{H\to H} \| R^{N-2} \|_{H\to H}$$

$$+\Big(\frac{3}{11} \left\| \tau A^{1/2}(I+i\tau A^{1/2})^{-1} \right\|_{H\to H} \| \widetilde{R} \|_{H\to H}$$

$$+\left\|(I+i\tau A^{1/2})^{-1}\right\|_{H\to H}\|A^{-1/2}\widetilde{J}_7\|_{H\to H}\Big)\|\widetilde{J}_1\|_{H\to H}\|\tilde{R}^{N-2}\|_{H\to H}\Big)\Big]$$

$$\times\Big[|\alpha|\Big(\frac{1}{2}\Big(\|\tau A^{1/2}\widetilde{J}_2\|_{H\to H}\|\tilde{R}^{N-1}\|_{H\to H}+\|\tau A^{1/2}J_2\|_{H\to H}\|R^{N-1}\|_{H\to H}\Big)$$

$$\times\tau\|A^{1/2}f_{2,2}\|_H+\frac{1}{2}\left\|\tau A^{1/2}\Big(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\Big)^{-1}\right\|_{H\to H}\tau\|A^{1/2}f_{N-1}\|_H$$

$$+\frac{1}{2}\Bigg(\sum_{s=2}^{N-2}\Big(\|\tilde{R}^{N-s}\|_{H\to H}+\|R^{N-s}\|_{H\to H}\Big)\|f_s-f_{s-1}\|_H+2\|f_{N-2}\|_H$$

$$+\Big(\|\tilde{R}^{N-1}\|_{H\to H}+\|R^{N-1}\|_{H\to H}\Big)\|f_1\|_H\Big)+\|A\varphi\|_H\Big]\Big\}$$

$$\leq M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\varphi\|+\|A^{1/2}\psi\|_H+\tau\|A^{1/2}f_{2,2}\|_H\right\}\qquad(3.55)$$

elde edilir. Şimdi (3.47), (3.48), (3.49) kestirimlerini elde edelim. İlk olarak formül (3.42), kararlılık kestirimleri (2.73), (3.39), (3.50), (3.51), (3.52), (3.53) den yararlanarak, üçgen eşitsizliği uygulanırsa $k\geq 2$ hali için

$$\left\|\frac{u_k+u_{k-1}}{2}\right\|_H\leq\frac{1}{2}\Big(\|J_1\|_{H\to H}\|R^{k-1}\|_{H\to H}+\|\widehat{J}_1\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H}\Big)\|\mu\|_H$$

$$+\frac{1}{2}\Big[\|\widetilde{J}_1\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H}+\|J_1\|_{H\to H}\|R^{k-1}\|_{H\to H}\Big]|i|\|A^{-1/2}\omega\|_H$$

$$+\frac{1}{2}\Big[\|\tau A^{1/2}\widehat{J}_2\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H}+\|\tau A^{1/2}J_2\|_{H\to H}\|R^{k-1}\|_{H\to H}\Big]\tau\|A^{-1/2}f_{2,2}\|_H$$

$$+\frac{1}{2}\left\|\tau A^{1/2}\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{-1/2}f_{k-1}\right\|_H+\frac{1}{2}\sum_{s=1}^{k-2}\Big[\left\|\tau A^{1/2}\widehat{J}_2\right\|_{H\to H}\left\|\tilde{R}^{k-1-s}\right\|_{H\to H}$$

$$+\|\tau A^{1/2}J_2\|_{H\to H}\|R^{k-1-s}\|_{H\to H}\Big]\tau\|A^{-1/2}f_s\|_H$$

$$\leq M\left\{\sum_{s=1}^{N-1}\left\|A^{-1/2}f_s\right\|_H\tau+\|\varphi\|_H+\left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{2,2}\right\|_H\right\} \tag{3.56}$$

kararlılık kestirimi elde edilir. İkinci olarak (3.42) formülüne, $A^{1/2}$ operatörü uygulanarak, $k\geq 2$ hali için

$$\left\|A^{1/2}\left(\frac{u_k+u_{k-1}}{2}\right)\right\|_H\leq\frac{1}{2}\big(\|J_1\|_{H\to H}\|R^{k-1}\|_{H\to H}$$

$$+\|\widetilde{J}_1\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H}\big)\|A^{1/2}\mu\|_H$$

$$+\frac{1}{2}\Big[\|\widetilde{J}_1\|_{H\to H}\|\tilde{R}^{k-1}\|_{H\to H}+\|J_1\|_{H\to H}\|R^{k-1}\|_{H\to H}\Big]|i|\|\omega\|_H$$

$$+\frac{1}{2}\Big[\left\|\tau A^{1/2}\widehat{J}_2\right\|_{H\to H}\left\|\tilde{R}^{k-1}\right\|_{H\to H}+\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\Big]\tau\left\|f_{2,2}\right\|_H$$

$$+\frac{1}{2}\left\|\tau A^{1/2}\left(I+\frac{\tau^2 A}{12}+\frac{\tau^4 A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|f_{k-1}\|_H$$

$$+\frac{1}{2}\sum_{s=1}^{k-2}\Big[\left\|\tau A^{1/2}\widehat{J}_2\right\|_{H\to H}\left\|\tilde{R}^{k-1-s}\right\|_{H\to H}+\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{k-1-s}\right\|_{H\to H}\Big]\tau\left\|f_s\right\|_H$$

$$\leq M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|A^{1/2}\varphi\|_H+\|\psi\|_H+\tau\|f_{2,2}\|_H\right\} \tag{3.57}$$

kararlılık kestirimi elde edilir. Daha sonra (3.42) ye Abel formülü uygulanarak

$$\frac{u_k + u_{k-1}}{2} = \frac{1}{2}\left[J_1 R^{k-1} + \widehat{J}_1 \tilde{R}^{k-1} \right]\mu + \frac{1}{2}\left[\widehat{J}_1 \tilde{R}^{k-1} - J_1 R^{k-1} \right] i A^{-1/2}\omega$$

$$+\frac{1}{2}\left[\widehat{J}_2 \tilde{R}^{k-1} - J_2 R^{k-1} \right]\tau^2 f_{2,2} + \frac{1}{2}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-1} \tau^2 f_{k-1}$$

$$+\frac{1}{2} A^{-1}\left(-\sum_{s=2}^{N-2} \left(\tilde{R}^{N-s} + R^{N-s} \right)\left(f_s - f_{s-1} \right) + 2 f_{N-2} - \left(\tilde{R}^{N-1} + R^{N-1} \right) f_1 \right), 2 \le k \le N \qquad (3.58)$$

formülü yazıldı. Buradan, (3.58) e, A operatörü ve üçgen eşitsizliği uygulanarak, (2.73), (3.39) kestirimlerinden yararlanılırsa, $k \ge 2$ hali için

$$\left\| A\left(\frac{u_k + u_{k-1}}{2} \right) \right\|_H \le \frac{1}{2}\left(\left\| J_1 \right\|_{H\to H} \left\| R^{k-1} \right\|_{H\to H} + \left\| \widehat{J}_1 \right\|_{H\to H} \left\| \tilde{R}^{k-1} \right\|_{H\to H} \right) \left\| A\mu \right\|_H$$

$$+ \frac{1}{2}\left[\left\| \widehat{J}_1 \right\|_{H\to H} \left\| \tilde{R}^{k-1} \right\|_{H\to H} + \left\| J_1 \right\|_{H\to H} \left\| R^{k-1} \right\|_{H\to H} \right] |i| \left\| A^{1/2}\omega \right\|_H$$

$$+\frac{1}{2}\left[\left\| \tau A^{1/2} \widehat{J}_2 \right\|_{H\to H} \left\| \tilde{R}^{k-1} \right\|_{H\to H} + \left\| \tau A^{1/2} J_2 \right\|_{H\to H} \left\| R^{k-1} \right\|_{H\to H} \right] \tau \left\| A^{1/2} f_{2,2} \right\|_H$$

$$+ \frac{1}{2} \left\| \tau A^{1/2} \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-1} \right\|_{H\to H} \tau \left\| A^{1/2} f_{k-1} \right\|_H$$

$$+ \frac{1}{2}\left(\sum_{s=2}^{N-2} \left(\left\| \tilde{R}^{N-s} \right\|_{H\to H} + \left\| R^{N-s} \right\|_{H\to H} \right) \left\| f_s - f_{s-1} \right\|_H \right.$$

$$\left. +2 \left\| f_{N-2} \right\|_H + \left(\left\| \tilde{R}^{N-1} \right\|_{H\to H} + \left\| R^{N-1} \right\|_{H\to H} \right) \left\| f_1 \right\|_H \right)$$

$$\le M \left\{ \sum_{s=2}^{N-1} \left\| f_s - f_{s-1} \right\|_H + \left\| f_1 \right\|_H + \left\| A\varphi \right\|_H + \left\| A^{1/2}\psi \right\|_H + \tau \left\| A^{1/2} f_{2,2} \right\|_H \right\} \qquad (3.59)$$

kararlılık kestirimi elde edilir. Böylece, Teorem 3.4 ün ispatı tamamlanmış olur. Şimdi Teorem 3.4 ün bazı sonuçlarını verelim. Önce karışık tip, lokal olmayan hiperbolik problem (3.29) ele alındı. Problem (3.29) un ayrışımı iki adımda ele alındı. İlk adım, 3.2 başlığındaki uygulamanın tamamen aynıdır. İkinci adımda (3.30) probleminin fark şeması

$$\begin{cases} \tau^{-2}\left(u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)\right)+\frac{5}{6}A_h^x u_k^h(x) \\ +\frac{1}{12}A_h^x\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)-\frac{1}{72}\tau^2\left(A_h^x\right)^2 u_k^h(x) \\ +\frac{\tau^2}{144}\left(A_h^x\right)^2\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)=f_k^h(x), \\ f_k^h(x)=\frac{5}{6}f^h(t_k,x)+\frac{1}{12}\left(f^h(t_{k+1},x)+f^h(t_{k-1},x)\right) \\ +\frac{\tau^2}{72}\left(-A_h^x f^h\left(t_k,x\right)+f_{tt}^h\left(t_k,x\right)\right) \\ -\frac{\tau^2}{144}\left(-A_h^x\left(f^h\left(t_{k+1},x\right)+f^h\left(t_{k-1},x\right)\right)\right. \\ \left.+f_{tt}^h\left(t_{k+1},x\right)+f_{tt}^h\left(t_{k-1},x\right)\right), x\in[0,1]_h, \\ t_k=k\tau, N\tau=1, 1\le k\le N-1, \\ u_0^h(x)=\alpha u_N^h(x)+\varphi^h(x), x\in[0,1]_h, \\ \left(I+\frac{\tau^2}{12}\left(A_h^x\right)+\frac{\tau^4}{144}\left(A_h^x\right)^2\right)\tau^{-1}\left(u_1^h(x)-u_0^h(x)\right) \\ +\frac{\tau}{2}\left(A_h^x\right)\varphi^h\left(x\right)-\tau f_{2,2}^h\left(x\right)=\beta\left(I-\frac{\tau^2}{12}\left(A_h^x\right)\right) \\ \times\left(\frac{1}{66\tau}\left(85u_N^h(x)-108u_{N-1}^h(x)+27u_{N-2}^h(x)-4u_{N-3}^h(x)\right)\right. \\ \left.+\frac{3\tau}{11}\left(f_N^h\left(x\right)-Au_N^h\left(x\right)\right)\right)+\left(I-\frac{\tau^2}{12}\left(A_h^x\right)\right)\psi^h(x), x\in[0,1]_h, \\ f_{2,2}^h\left(x\right)=\frac{1}{2}f^h\left(0,x\right)+\frac{\tau}{6}f_t^h\left(0,x\right)+\frac{\tau^2}{24}f_{tt}^h\left(0,x\right). \end{cases} \tag{3.60}$$

yazıldı.

Teorem 3.5 τ ve h yeterince küçük sayılar olsun. Bu takdirde fark şeması (3.60) ın çözümü için aşağıdaki kararlılık kestirimleri sağlanır;

$$\max_{1\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{L_{2h}}+\max_{1\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^1}$$

$$\le M_1\left[\max_{1\le k\le N-1}\|f_k^h\|_{L_{2h}}+\|\psi^h\|_{L_{2h}}+\|\varphi^h\|_{W_{2h}^1}+\tau\|f_{2,2}^h\|_{L_{2h}}\right],$$

$$\max_{1\le k\le N-1}\|\tau^{-2}(u_{k+1}^h-u_{k-1}^h)\|_{W_{2h}^1}+\max_{0\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^2}$$

$$\leq M_1\left[\left\|f_1^h\right\|_{L_{2h}}+\max_{2\leq k\leq N-1}\left\|\tau^{-1}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}+\left\|\psi^h\right\|_{W_{2h}^1}+\left\|\varphi^h\right\|_{W_{2h}^2}+\tau\left\|f_{2,2}^h\right\|_{W_{2h}^1}\right].$$

Burada M_1 sabiti, $\tau,\ h,\ \varphi^h(x),\ \psi^h(x),\ f_{2,2}^h$ ve $f_k^h\ (1\leq k<N)$ ifadelerinden bağımsızdır.

Teorem 3.5 in ispatı soyut (abstract) Teorem 3.4 ün ispatına ve (2.65) de verilen A_h^x fark operatörünün simetri özeliğine bağlıdır.

Problem (3.32) yi ele alalım. Problem (3.32) nin ayrışımı iki adımda yapılabilir. İlk adım, 3.2 başlığındaki adımlarla tamamen aynıdır.

İkinci adımda, problem (3.33) nin fark şeması

$$\begin{cases}\tau^{-2}\left(u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)\right)+\frac{5}{6}A_h^x u_k^h(x)\\ +\frac{1}{12}A_h^x\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)-\frac{1}{72}\tau^2\left(A_h^x\right)^2u_k^h(x)\\ +\frac{\tau^2}{144}\left(A_h^x\right)^2\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)=f_k^h(x),\\ f_k^h(x)=\frac{5}{6}f^h(t_k,x)+\frac{1}{12}\left(f^h(t_{k+1},x)+f^h(t_{k-1},x)\right)\\ +\frac{\tau^2}{72}\left(-A_h^xf^h\left(t_k,x\right)+f_{tt}^h\left(t_k,x\right)\right)\\ -\frac{\tau^2}{144}\left(-A_h^x\left(f^h\left(t_{k+1},x\right)+f^h\left(t_{k-1},x\right)\right)\right.\\ \left.+f_{tt}^h\left(t_{k+1},x\right)+f_{tt}^h\left(t_{k-1},x\right)\right),x\in\Omega_h,\\ t_k=k\tau,1\leq k\leq N-1,N\tau=1,\\ u_0^h(x)=\alpha u_N^h(x)+\varphi^h(x),x\in\widehat{\Omega}_h,\\ \left(I+\frac{\tau^2}{12}\left(A_h^x\right)+\frac{\tau^4}{144}\left(A_h^x\right)^2\right)\tau^{-1}\left(u_1^h(x)-u_0^h(x)\right)\\ +\frac{\tau}{2}\left(A_h^x\right)\varphi^h\left(x\right)-\tau f_{2,2}^h\left(x\right)=\beta\left(I-\frac{\tau^2}{12}\left(A_h^x\right)\right)\\ \times\left(\frac{1}{66\tau}\left(85u_N^h(x)-108u_{N-1}^h(x)+27u_{N-2}^h(x)-4u_{N-3}^h(x)\right)\right.\\ \left.+\frac{3\tau}{11}\left(f_N^h\left(x\right)-Au_N^h\left(x\right)\right)\right)+\left(I-\frac{\tau^2}{12}\left(A_h^x\right)\right)\psi^h(x),x\in\widehat{\Omega}_h,\\ f_{2,2}^h\left(x\right)=\frac{1}{2}f^h\left(0,x\right)+\frac{\tau}{6}f_t^h\left(0,x\right)+\frac{\tau^2}{24}f_{tt}^h\left(0,x\right)\end{cases}\tag{3.61}$$

yazıldı.

Teorem 3.6 τ ve $|h|$ yeterince küçük sayılar olsun. Bu takdirde fark şeması (3.61) in çözümü için, aşağıdaki kararlılık kestirimlerini sağlanır;

$$\max_{1\le k\le N}\left\| \frac{u_k^h + u_{k-1}^h}{2} \right\|_{L_{2h}} + \max_{1\le k\le N}\left\| \frac{u_k^h + u_{k-1}^h}{2} \right\|_{W_{2h}^1}$$

$$\le M_1\left[\max_{1\le k\le N-1} \| f_k^h \|_{L_{2h}} + \|\psi^h\|_{L_{2h}} + \|\varphi^h\|_{W_{2h}^1} + \tau \| f_{2,2}^h \|_{L_{2h}} \right],$$

$$\max_{1\le k\le N-1} \|\tau^{-2}(u_{k+1}^h - u_{k-1}^h)\|_{W_{2h}^1} + \max_{0\le k\le N}\left\| \frac{u_k^h + u_{k-1}^h}{2} \right\|_{W_{2h}^2}$$

$$\le M_1\left[\|f_1^h\|_{L_{2h}} + \max_{2\le k\le N-1} \|\tau^{-1}(f_k^h - f_{k-1}^h)\|_{L_{2h}} \right.$$

$$\left. + \|\psi^h\|_{W_{2h}^1} + \|\varphi^h\|_{W_{2h}^2} + \tau \|f_{2,2}^h\|_{W_{2h}^1} \right].$$

Burada M_1 sabiti, $\tau,\ h,\ \varphi^h(x),\ \psi^h(x),\ f_{2,2}^h$ ve f_k^h $(1\le k < N)$ ifadelerinden bağımsızdır.

Teorem 3.6 nın ispatı abstract (soyut) Teorem 3.4 ün ispatına, (2.69) da verilen A_h^x fark operatörünün simetri özeliğine ve Teorem 2.4 e bağlıdır.

4. SAYISAL SONUÇLAR

4.1 Problem

Bu bölümde hiperbolik tip Cauchy problemi

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = 2e^{-t}\sin x, \\ 0 < t < 1, 0 < x < \pi, \\ u(0,x) = \varphi(x), \\ \varphi(x) = \sin x, 0 \le x \le \pi, \\ u_t(0,x) = \psi(x), \\ \psi(x) = -\sin x, 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0,\ 0 \le t \le 1 \end{cases} \tag{4.1}$$

ve hiperbolik tip lokal olmayan sınır değer problemi

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = 2e^{-t}\sin x, \\ 0 < t < 1, 0 < x < \pi, \\ u(0,x) - \frac{1}{2}u(1,x) = \varphi(x), \\ \varphi(x) = \left(1 - \frac{1}{2}e^{-1}\right)\sin x, 0 \le x \le \pi, \\ u_t(0,x) - \frac{1}{2}u_t(1,x) = \psi(x), \\ \psi(x) = -\left(1 - \frac{1}{2}e^{-1}\right)\sin x, 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0,\ 0 \le t \le 1 \end{cases} \tag{4.2}$$

ele alındı. (4.1) ve (4.2) nin kesin çözümü $u(t,x) = e^{-t}\sin x$ tir.

Yukarıdaki problemlerin yaklaşık çözümlerini elde ederken,

$$[0,1]_\tau \times [0,\pi]_h = \{(t_k, x_n):\ t_k = k\tau, 0 \le k \le N, N\tau = 1, x_n = nh,\ 0 \le n \le M, Mh = \pi\}$$

şeklinde tanımlanan, τ ve h a bağlı grid noktalar kümesi ailesi $[0,1]_\tau \times [0,\pi]_h$ ve sırasıyla üçüncü ve dördüncü mertebeden doğruluk fark şemaları kullanıldı. Fark Çözümler için n e bağlı matris katsayılı Modifiye Edilmiş Gauss Eliminasyon metodundan yararlanıldı. Teorik sonuçlar Matlab programı kullanılarak elde edilen nümerik sonuçlarla desteklendi.

4.2 Hiperbolik Tip Cauchy Probleminin Yaklaşık Çözümü İçin Zamana Bağlı Üçüncü Mertebeden Doğruluk Fark Şeması

Bu kısımda, aşağıdaki basit yaklaşım formüller

$$\frac{u(x_{n+1})-2u(x_n)+u(x_{n-1})}{h^2}-u''(x_n)=O(h^2), \tag{4.3}$$

$$\frac{u(x_{n+2})-4u(x_{n+1})+6u(x_n)-4u(x_{n-1})+u(x_{n-2})}{h^4}-u^{(iv)}(x_n)=O(h^2), \tag{4.4}$$

$$\frac{2u(0)-5u(h)+4u(2h)-u(3h)}{h^2}-u''(0)=O(h^2), \tag{4.5}$$

$$\frac{2u(1)-5u(1-h)+4u(1-2h)-u(1-3h)}{h^2}-u''(1)=O(h^2), \tag{4.6}$$

ve fark şeması (2.24) kullanılarak Cauchy problemi (4.1) in zamana yani t ye bağlı üçüncü mertebeden, uzay boyutuna yani x e bağlı ikinci mertebeden yaklaşık çözümü için

$$
\begin{cases}
\dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\frac{2}{3}\left(\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}\right) \\
-\frac{1}{6}\left(\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+\dfrac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\right) \\
+\frac{\tau^2}{12}\left(\dfrac{u_{n+2}^{k+1}-4u_{n+1}^{k+1}+6u_n^{k+1}-4u_{n-1}^{k+1}+u_{n-2}^{k+1}}{h^4}\right)=\varphi_n^k, \\
\varphi_n^k=\left(\frac{4}{3}\exp(-t_k)+\frac{1}{3}\left(\exp(-t_{k+1})+\exp(-t_{k-1})\right)\right)\sin(x_n), \\
t_k=k\tau,\ 1\le k\le N-1, N\tau=1, x_n=nh, 2\le n\le M-2, Mh=\pi, \\
u_n^0=\varphi_n^0, \varphi_n^0=\sin(x_n), 0\le n\le M, \\
u_n^1-u_n^0-\frac{\tau^2}{12}\left(\dfrac{\left(u_{n+1}^1-u_{n+1}^0\right)-2\left(u_n^1-u_n^0\right)+\left(u_{n-1}^1-u_{n-1}^0\right)}{h^2}\right)+\frac{\tau^4}{144} \\
\times\left(\dfrac{u_{n+2}^1-u_{n+2}^0-4\left(u_{n+1}^1-u_{n+1}^0\right)+6\left(u_n^1-u_n^0\right)-4\left(u_{n-1}^1-u_{n-1}^0\right)+u_{n-2}^1-u_{n-2}^0}{h^4}\right) \\
=\varphi_n^N, \varphi_n^N=-\frac{\tau^2}{2}A\sin(x_n)+\tau\left(I-\frac{\tau^2}{12}A\right)\left(-\sin(x_n)\right)+f_{1,1}\tau, 2\le n\le M-2, \\
f_{1,1}=\left(f(0)+\left(-f(0)+\tau f'(0)\right)\frac{1}{2}-2f'(0)\frac{\tau}{6}\right) \\
u_0^k=u_M^k=0, 0\le k\le N, \\
4u_2^k-5u_1^k, u_{M-3}^k=4u_{M-2}^k-5u_{M-1}^k, 0\le k\le N
\end{cases}
\tag{4.7}
$$

problemi elde edildi. Problem (4.7), $(N+1)\times(N+1)$ boyutlu lineer denklem sistemi olarak yazılabilir. (4.7) yeniden düzenlenerek matris formda yazılacak olursa,

$$
\begin{cases}
\left(\frac{\tau^2}{12h^4}\right)u_{n+2}^{k+1}+\left(-\frac{1}{6h^2}-\frac{\tau^2}{3h^4}\right)u_{n+1}^{k+1}+\left(-\frac{2}{3h^2}\right)u_{n+1}^k \\
+\left(-\frac{1}{6h^2}\right)u_{n+1}^{k-1}+\left(\frac{1}{\tau^2}+\frac{1}{3h^2}+\frac{\tau^2}{2h^4}\right)u_n^{k+1}+\left(-\frac{2}{\tau^2}+\frac{4}{3h^2}\right)u_n^k \\
+\left(\frac{1}{\tau^2}+\frac{1}{3h^2}\right)u_n^{k-1}+\left(-\frac{1}{6h^2}-\frac{\tau^2}{3h^4}\right)u_{n-1}^{k+1}+\left(-\frac{2}{3h^2}\right)u_{n-1}^k \\
+\left(-\frac{1}{6h^2}\right)u_{n-1}^{k-1}+\left(\frac{\tau^2}{12h^4}\right)u_{n-2}^{k+1}=\varphi_n^k, 1\le k\le N-1, 2\le n\le M-2, \\
u_n^0=\varphi_n^0=\sin(x_n), 0\le n\le M, \\
\left(\frac{\tau^4}{144h^4}\right)u_{n+2}^1+\left(-\frac{\tau^4}{144h^4}\right)u_{n+2}^0+\left(-\frac{\tau^4}{36h^4}-\frac{\tau^2}{12h^2}\right)u_{n+1}^1 \\
+\left(\frac{\tau^4}{36h^4}+\frac{\tau^2}{12h^2}\right)u_{n+1}^0+\left(1+\frac{\tau^2}{6h^2}+\frac{\tau^4}{24h^4}\right)u_n^1+\left(-1-\frac{\tau^2}{6h^2}-\frac{\tau^4}{24h^4}\right)u_n^0 \\
+\left(-\frac{\tau^4}{36h^4}-\frac{\tau^2}{12h^2}\right)u_{n-1}^1+\left(\frac{\tau^4}{36h^4}+\frac{\tau^2}{12h^2}\right)u_{n-1}^0+\left(\frac{\tau^4}{144h^4}\right)u_{n-2}^1 \\
+\left(-\frac{\tau^4}{144h^4}\right)u_{n-2}^0=\varphi_n^N, \varphi_n^N=\left(\tau-\frac{\tau^2}{3}\right)\sin(x_n), 2\le n\le M-2, \\
u_0^k=u_M^k=0, 0\le k\le N, \\
4u_2^k-5u_1^k, u_{M-3}^k=4u_{M-2}^k-5u_{M-1}^k, 0\le k\le N
\end{cases}
\tag{4.8}
$$

elde edilir. Düzenlemeden sonra lineer sistem

$$\begin{cases} AU_{n+2} + BU_{n+1} + CU_n + DU_{n-1} + EU_{n-2} = R\varphi_n, \ \ 2 \le n \le M-2, \\ U_0 = U_M = \widehat{0}, \ U_3^k = 4U_2^k - 5U_1^k, \ U_{M-3}^k = 4U_{M-2}^k - 5U_{M-1}^k, \ 0 \le k \le N \end{cases}$$

şeklinde oluşur. Burada

$$\varphi_n^k = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1} ,0 \le k \le N, \ \varphi_n^0 = \sin(x_n), \ \varphi_n^N = \left(\tau - \frac{\tau^2}{3}\right)\sin(x_n),$$

$$\varphi_n^k = \left(\tfrac{4}{3}\exp(-t_k) + \tfrac{1}{3}\left(\exp(-t_{k+1}) + \exp(-t_{k-1})\right)\right)\sin(x_n), \ 1 \le k \le N-1$$

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ x & 0 & \cdots & 0 & 0 & 0 \\ 0 & x & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x & 0 \\ a & aa & \cdots & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ v & vv & vvv & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & v & vv & vvv & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & v & vv & vvv \\ b & bb & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ m & mm & mmm & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & m & mm & mmm & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & m & mm & mmm \\ c & cc & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)} , D = B,\ E = A,$$

$$R=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\text{ve } U_s^k = \begin{bmatrix} u_s^0 \\ u_s^1 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1}, 0 \le k \le N, s = n-2, n-1, n, n+1, n+2$$

dir. A, B ve C matrislerinin elemanları için

$$x = \frac{\tau^2}{12h^4}, vvv = -\frac{1}{6h^2} - \frac{\tau^2}{3h^4}, vv = -\frac{2}{3h^2}, v = -\frac{1}{6h^2}, mmm = \frac{1}{\tau^2} + \frac{1}{3h^2} + \frac{\tau^2}{2h^4},$$

$$mm = -\frac{2}{\tau^2} + \frac{4}{3h^2}, m = \frac{1}{\tau^2} + \frac{1}{3h^2}, a = -\frac{\tau^2}{144h^4}, aa = \frac{\tau^2}{144h^4}, b = \frac{\tau^4}{36h^4} + \frac{\tau^2}{12h^2},$$

$$bb = -\frac{\tau^4}{36h^4} - \frac{\tau^2}{12h^2}, c = -1 - \frac{\tau^2}{6h^2} - \frac{\tau^4}{24h^4}, cc = 1 + \frac{\tau^2}{6h^2} + \frac{\tau^4}{24h^4}$$

gösterimleri kullanılmıştır. Fark denklemi (4.7) nin çözümünde, Modifiye Edilmiş Gauss Eliminasyon metodu kullanıldığından,

$$U_n = \alpha_{n+1} U_{n+1} + \beta_{n+1} U_{n+2} + \gamma_{n+1}, n = M - 2, \ldots 2, 1, 0 \tag{4.9}$$

formunda, $\alpha_j, \beta_j \quad (j = 1, \cdots, M)$ lerin $(N+1) \times (N+1)$ kare matrisler ve γ_j lerin $(N+1) \times 1$ kolon matrisleri olduğu çözüm elde edildi. α_{n+1}, β_{n+1}, γ_{n+1} $(n = 2 : M - 2)$ için

$$\begin{cases} \beta_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(A), \\ \alpha_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(B + D\beta_n + E\alpha_{n-1}\beta_n), \\ \gamma_{n+1} = +(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1} \\ \times(R\varphi_n - D\gamma_n - E\alpha_{n-1}\gamma_n - E\gamma_{n-1}), \end{cases} \tag{4.10}$$

formülleri elde edildi. Burada $\alpha_1,\ \beta_1,\ \gamma_1,\ \alpha_2,\ \beta_2,\ \gamma_2$

$$\alpha_1 = \begin{bmatrix} 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ \vdots & \ldots & \vdots \\ 0 & \ldots & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \beta_1 = \begin{bmatrix} 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ \vdots & \ldots & \vdots \\ 0 & \ldots & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\gamma_1 = \gamma_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(N+1)\times 1}, \quad \alpha_2 = \begin{bmatrix} \frac{4}{5} & 0 & \ldots & 0 \\ 0 & \frac{4}{5} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \frac{4}{5} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\beta_2 = \begin{bmatrix} -\frac{1}{5} & 0 & \ldots & 0 \\ 0 & -\frac{1}{5} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & -\frac{1}{5} \end{bmatrix}_{(N+1)\times(N+1)},$$

$U_M = \widehat{0}$ dir. Ayrıca

$$\begin{cases} U_{M-2} = \alpha_{M-1} U_{M-1} + \beta_{M-1}, \\ U_{M-3} = \alpha_{M-2} U_{M-2} + \beta_{M-2} U_{M-1} + \gamma_{M-2}, \\ U_{M-3} = 4U_{M-2} - 5U_{M-1}, \end{cases}$$

formülleri kullanılarak

$(4I - \alpha_{M-2})U_{M-2} + (-5I - \beta_{M-2})U_{M-1} = \gamma_{M-2}$ veya

$(4I - \alpha_{M-2})(\alpha_{M-1}U_{M-1} + \gamma_{M-1}) - (5I + \beta_{M-2})U_{M-1} = \gamma_{M-2}$ formülü yazılabilir.

Buradan da $U_{M-1} = [(\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1}]^{-1}[(4I - \alpha_{M-2})\gamma_{M-1} - \gamma_{M-2}]$ bulunur.

Böylece, lineer sistemden, formüllerden, matrislerden ve üçüncü mertebeden yaklaşımlı fark şemasından yararlanarak, Cauchy problemi (4.1) in yaklaşık çözümü elde edilir. Ayrıca, Matlab programı kullanılarak değişik M ve N değerleri için elde edilen sayısal sonuçlar hata analizi başlığında, yazılmış olan program kodları ise ekler bölümünde sunulmuştur.

4.3 Hiperbolik Tip Cauchy Probleminin Yaklaşık Çözümü İçin Zamana Bağlı Dördüncü Mertebeden Doğruluk Fark Şeması

Bu kısımda, yaklaşım formülleri (4.3), (4.4), (4.5), (4.6), ve fark şeması (2.72) kullanılarak, Cauchy problemi (4.1) in zamana yani t ye bağlı dördüncü mertebeden, uzay boyutuna yani x e bağlı ikinci mertebeden yaklaşık çözümü için, fark şeması (4.11) elde edildi.

$$
\begin{cases}
\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\frac{5}{6}\left(\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}\right) \\
-\frac{1}{12}\left(\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\right) \\
-\frac{\tau^2}{72}\left(\frac{u_{n+2}^k-4u_{n+1}^k+6u_n^k-4u_{n-1}^k+u_{n-2}^k}{h^4}\right) \\
+\frac{\tau^2}{144}\left(\frac{u_{n+2}^{k+1}-4u_{n+1}^{k+1}+6u_n^{k+1}-4u_{n-1}^{k+1}+u_{n-2}^{k+1}}{h^4}\right. \\
\left.+\frac{u_{n+2}^{k-1}-4u_{n+1}^{k-1}+6u_n^{k-1}-4u_{n-1}^{k-1}+u_{n-2}^{k-1}}{h^4}\right)=\varphi_n^k, \\
\varphi_n^k=\left(\frac{5}{3}(\exp(-t_k))+\frac{1}{6}(\exp(-t_{k+1})+\exp(-t_{k-1}))\right)\sin(x_n), \\
t_k=k\tau,\ 1\le k\le N-1, N\tau=1, x_n=nh, 1\le n\le M-1, Mh=\pi, \\
u_n^0=\varphi_n^0, \varphi_n^0=\sin(x_n), 0\le n\le M, \\
u_n^1-u_n^0-\frac{\tau^2}{12}\left(\frac{(u_{n+1}^1-u_{n+1}^0)-2(u_n^1-u_n^0)+(u_{n-1}^1-u_{n-1}^0)}{h^2}\right) \\
+\frac{\tau^4}{144}\left(\frac{u_{n+2}^1-u_{n+2}^0-4\left(u_{n+1}^1-u_{n+1}^0\right)+6\left(u_n^1-u_n^0\right)-4\left(u_{n-1}^1-u_{n-1}^0\right)+u_{n-2}^1-u_{n-2}^0}{h^4}\right)=\varphi_n^N, \\
\varphi_n^N=-\frac{\tau^2}{2}A\sin(x_n)+\tau\left(I-\frac{\tau^2}{12}A\right)(-\sin(x_n))+\tau f_{2,2} \\
f_{2,2}=\left(I-\frac{\tau^2A}{12}\right)f(0)\tau-\left(\left(I-\frac{5\tau^2A}{12}\right)-\tau f^{'}(0)\right)\frac{\tau}{2} \\
+(-A\tau f(0)-2f^{'}(0)+\tau f^{''}(0))\frac{\tau^2}{6}+(Af(0)-3f^{''}(0))\frac{\tau^3}{24}, 2\le n\le M-2, \\
u_0^k=u_M^k=0, 0\le k\le N, \\
u_3^k=4u_2^k-5u_1^k, u_{M-3}^k=4u_{M-2}^k-5u_{M-1}^k, 0\le k\le N.
\end{cases}
\tag{4.11}
$$

Problem (4.11), $(N+1)\times(N+1)$ boyutlu lineer denklem sistemidir ve yeniden düzenlenerek matris formda yazılacak olursa fark şeması (4.12) elde edilir.

$$
\begin{cases}
\left(\frac{\tau^2}{144h^4}\right)u_{n+2}^{k+1}+\left(-\frac{\tau^2}{72h^4}\right)u_{n+2}^{k}+\left(\frac{\tau^2}{144h^4}\right)u_{n+2}^{k-1}+\left(-\frac{1}{12h^2}-\frac{4\tau^2}{144h^4}\right)u_{n+1}^{k+1}\\
+\left(-\frac{5}{6h^2}+\frac{4\tau^2}{72h^4}\right)u_{n+1}^{k}+\left(-\frac{1}{12h^2}-\frac{4\tau^2}{144h^4}\right)u_{n+1}^{k-1}+\left(\frac{1}{\tau^2}+\frac{1}{6h^2}+\frac{\tau^2}{24h^4}\right)u_{n}^{k+1}\\
+\left(-\frac{2}{\tau^2}+\frac{5}{3h^2}-\frac{\tau^2}{12h^4}\right)u_{n}^{k}+\left(\frac{1}{\tau^2}+\frac{1}{6h^2}+\frac{\tau^2}{12h^4}\right)u_{n}^{k-1}+\left(-\frac{1}{12h^2}-\frac{4\tau^2}{144h^4}\right)u_{n-1}^{k+1}\\
+\left(-\frac{5}{6h^2}+\frac{4\tau^2}{72h^4}\right)u_{n-1}^{k}+\left(-\frac{1}{12h^2}-\frac{4\tau^2}{144h^4}\right)u_{n-1}^{k-1}+\left(\frac{\tau^2}{144h^4}\right)u_{n-2}^{k+1}\\
+\left(-\frac{\tau^2}{72h^4}\right)u_{n-2}^{k}+\left(\frac{\tau^2}{144h^4}\right)u_{n-2}^{k-1}=\varphi_n^k, 1\le k\le N-1, 2\le n\le M-2,\\
u_n^0=\varphi_n^0=\sin(x_n), 0\le n\le M,\\
\left(\frac{\tau^4}{144h^4}\right)u_{n+2}^{1}+\left(-\frac{\tau^4}{144h^4}\right)u_{n+2}^{0}+\left(-\frac{\tau^4}{36h^4}-\frac{\tau^2}{12h^2}\right)u_{n+1}^{1}\\
+\left(\frac{\tau^4}{36h^4}+\frac{\tau^2}{12h^2}\right)u_{n+1}^{0}+\left(1+\frac{\tau^2}{6h^2}+\frac{\tau^4}{24h^4}\right)u_{n}^{1}+\left(-1-\frac{\tau^2}{6h^2}-\frac{\tau^4}{24h^4}\right)u_{n}^{0}\\
+\left(-\frac{\tau^4}{36h^4}-\frac{\tau^2}{12h^2}\right)u_{n-1}^{1}+\left(\frac{\tau^4}{36h^4}+\frac{\tau^2}{12h^2}\right)u_{n-1}^{0}+\left(\frac{\tau^4}{144h^4}\right)u_{n-2}^{1}\\
+\left(-\frac{\tau^4}{144h^4}\right)u_{n-2}^{0}=\varphi_n^N,\\
\varphi_n^N=\left(-\tau+\frac{\tau^2}{2}-\frac{\tau^3}{4}+\frac{\tau^4}{12}-\frac{\tau^5}{36}+\frac{\tau^6}{16}-\frac{\tau^7}{108}-\frac{\tau^8}{864}\right)\sin(x_n), 2\le n\le M-2,\\
u_0^k=u_M^k=0, 0\le k\le N,\\
u_3^k=4u_2^k-5u_1^k, u_{M-3}^k=4u_{M-2}^k-5u_{M-1}^k, 0\le k\le N.
\end{cases}
\tag{4.12}
$$

Düzenlemeden sonra lineer sistem

$$
\begin{cases}
AU_{n+2}+BU_{n+1}+CU_n+DU_{n-1}+EU_{n-2}=R\varphi_n,\ 2\le n\le M-2,\\
U_0=U_M=\widehat{0},\ U_3^k=4U_2^k-5U_1^k,\ U_{M-3}^k=4U_{M-2}^k-5U_{M-1}^k,\ 0\le k\le N
\end{cases}
$$

elde edilir. Burada, $\varphi_n^k=\begin{bmatrix}\varphi_n^0\\ \varphi_n^1\\ \vdots\\ \varphi_n^N\end{bmatrix}_{(N+1)\times 1}$, $0\le k\le N$,

$$
\varphi_n^0=\sin(x_n),\ \varphi_n^N=\left(-\tau+\frac{\tau^2}{2}-\frac{\tau^3}{4}+\frac{\tau^4}{12}-\frac{\tau^5}{36}+\frac{\tau^6}{16}-\frac{\tau^7}{108}-\frac{\tau^8}{864}\right)\sin(x_n),
$$

$$
\varphi_n^k=\left(\tfrac{5}{3}\left(\exp(-t_k)\right)+\tfrac{1}{6}\left(\exp(-t_{k+1})+\exp(-t_{k-1})\right)\right)\sin(x_n)\sin(x_n),\ 1\le k\le N-1,
$$

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ x & y & x & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & x & y & x & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & x & y & x \\ a & aa & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ v & w & v & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & v & w & v & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & v & w & v \\ b & bb & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ m & n & m & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & m & n & m & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & m & n & m \\ c & cc & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, D=B,\ E=A,$$

$$R=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

ve

$$U_s^k=\begin{bmatrix} u_s^0 \\ u_s^1 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1}, 0\le k\le N, s=n-2, n-1, n, n+1, n+2$$

dir. A, B ve C matrislerinin elemanları için

$$x=\frac{\tau^2}{144h^4},\ y=-\frac{\tau^2}{72h^4},\ v=-\frac{1}{12h^2}-\frac{4\tau^2}{144h^4},\ w=-\frac{5}{6h^2}+\frac{4\tau^2}{72h^4},$$

$$m=\frac{1}{\tau^2}+\frac{1}{6h^2}+\frac{\tau^2}{24h^4},\ n=-\frac{2}{\tau^2}+\frac{5}{3h^2}-\frac{\tau^2}{12h^4},\ a=-\frac{\tau^2}{144h^4},$$

$$aa=\frac{\tau^2}{144h^4},\ b=\frac{\tau^4}{36h^4}+\frac{\tau^2}{12h^2},\ bb=-\frac{\tau^4}{36h^4}-\frac{\tau^2}{12h^2},$$

$$c=-1-\frac{\tau^2}{6h^2}-\frac{\tau^4}{24h^4},\ cc=1+\frac{\tau^2}{6h^2}+\frac{\tau^4}{24h^4}$$

gösterimleri kullanılmıştır. Fark şeması (4.11) in çözümünde kullanılan yöntem bir önceki başlıkta sunulan yöntemle tamamen aynıdır. Böylece, lineer sistemden, formüllerden, matrislerden ve dördüncü mertebeden yaklaşımlı fark şemasından yararlanarak, Cauchy problemi (4.1) in yaklaşık çözümü elde edilir. Ayrıca, Matlab programı kullanılarak değişik M ve N değerleri için elde edilen sayısal sonuçlar hata analizi başlığında, yazılmış olan program kodları ise ekler bölümünde sunulmuştur.

4.4 Lokal Olmayan Hiperbolik Tip Sınır Değer Probleminin Yaklaşık Çözümü İçin Zamana Bağlı Üçüncü Mertebeden Doğruluk Fark Şeması

Bu kısımda, yaklaşım formülleri (4.3), (4.4), (4.5), (4.6),

$$\frac{11u(1)-18u(1-\tau)+9u(1-2\tau)-2u(1-3\tau)}{6\tau}-u'(1)=O(\tau^3), \tag{4.13}$$

ve fark şeması (3.2) kullanılarak, lokal olmayan sınır değer problemi (4.2) nin zamana yani t ye bağlı üçüncü mertebeden, uzay boyutuna yani x e bağlı ikinci mertebeden yaklaşık çözümü için fark şeması

$$
\begin{cases}
\dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\frac{2}{3}\left(\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}\right) \\
-\frac{1}{6}\left(\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+\dfrac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\right) \\
+\frac{\tau^2}{12}\left(\dfrac{u_{n+2}^k-4u_{n+1}^k+6u_n^k-4u_{n-1}^k+u_{n-2}^k}{h^4}\right)=\varphi_n^k, \\
\varphi_n^k=\left(\frac{4}{3}\exp(-t_k)+\frac{1}{3}\left(\exp(-t_{k+1})+\exp(-t_{k-1})\right)\right)\sin(x_n), \\
t_k=k\tau,\ 1\le k\le N-1, N\tau=1, x_n=nh, 2\le n\le M-2, Mh=\pi, \\
u_n^0-\frac{1}{2}u_n^N=\varphi_n^0, \varphi_n^0=\left(1-\frac{1}{2}e^{-1}\right)\sin(x_n), 0\le n\le M, \\
u_n^1-u_n^0-\frac{\tau^2}{12}\left(\dfrac{\left(u_{n+1}^1-u_{n+1}^0\right)-2\left(u_n^1-u_n^0\right)+\left(u_{n-1}^1-u_{n-1}^0\right)}{h^2}\right)+\frac{\tau^4}{144} \\
\times\left(\dfrac{u_{n+2}^1-u_{n+2}^0-4\left(u_{n+1}^1-u_{n+1}^0\right)+6\left(u_n^1-u_n^0\right)-4\left(u_{n-1}^1-u_{n-1}^0\right)+u_{n-2}^1-u_{n-2}^0}{h^4}\right) \\
+\frac{1}{2}\left(\dfrac{11u_n^N-18u_n^{N-1}+9u_n^{N-2}-2u_n^{N-3}}{6}\right)=\varphi_n^N, 2\le n\le M-2, \\
\varphi_n^N=-\frac{\tau^2}{2}A\sin(x_n)+\tau\left(I-\frac{\tau^2}{12}A\right)\left(-\sin(x_n)\right)+f_{1,1}\tau+\frac{1}{2}\tau e^{-1}\sin(x_n), \\
f_{1,1}=\left(f(0)+\left(-f(0)+\tau f'(0)\right)\frac{1}{2}-2f'(0)\frac{\tau}{6}\right), \\
u_0^k=u_M^k=0,\ 0\le k\le N, \\
u_3^k=4u_2^k-5u_1^k,\ u_{M-3}^k=4u_M^k-5u_{M-1}^k
\end{cases}
\tag{4.14}
$$

elde edildi. Problem $(N+1)\times(N+1)$ boyutlu lineer denklem sistemi olarak yazılabilir. Sistem yeniden düzenlenerek matris formda yazılacak olursa fark şeması (4.15) elde edilir.

$$
\begin{cases}
\left(\frac{\tau^2}{12h^4}\right)u_{n+2}^{k+1} + \left(-\frac{1}{6h^2} - \frac{\tau^2}{3h^4}\right)u_{n+1}^{k+1} + \left(-\frac{2}{3h^2}\right)u_{n+1}^{k} \\
+\left(-\frac{1}{6h^2}\right)u_{n+1}^{k-1} + \left(\frac{1}{\tau^2} + \frac{1}{3h^2} + \frac{\tau^2}{2h^4}\right)u_{n}^{k+1} + \left(-\frac{2}{\tau^2} + \frac{4}{3h^2}\right)u_{n}^{k} \\
+\left(\frac{1}{\tau^2} + \frac{1}{3h^2}\right)u_{n}^{k-1} + \left(-\frac{1}{6h^2} - \frac{\tau^2}{3h^4}\right)u_{n-1}^{k+1} + \left(-\frac{2}{3h^2}\right)u_{n-1}^{k} \\
+\left(-\frac{1}{6h^2}\right)u_{n-1}^{k-1} + \left(\frac{\tau^2}{12h^4}\right)u_{n-2}^{k+1} = \varphi_n^k, 1 \le k \le N-1, 2 \le n \le M-2, \\
u_n^0 - \frac{1}{2}u_n^N = \varphi_n^0, \varphi_n^0 = (1 - \frac{1}{2}e^{-1})\sin(x_n), 0 \le n \le M, \\
\left(\frac{\tau^4}{144h^4}\right)u_{n+2}^{1} + \left(-\frac{\tau^4}{144h^4}\right)u_{n+2}^{0} + \left(-\frac{\tau^4}{36h^4} - \frac{\tau^2}{12h^2}\right)u_{n+1}^{1} \\
+\left(\frac{\tau^4}{36h^4} + \frac{\tau^2}{12h^2}\right)u_{n+1}^{0} + \left(1 + \frac{\tau^2}{6h^2} + \frac{\tau^4}{24h^4}\right)u_{n}^{1} + \left(-1 - \frac{\tau^2}{6h^2} - \frac{\tau^4}{24h^4}\right)u_{n}^{0} \\
+\left(-\frac{\tau^4}{36h^4} - \frac{\tau^2}{12h^2}\right)u_{n-1}^{1} + \left(\frac{\tau^4}{36h^4} + \frac{\tau^2}{12h^2}\right)u_{n-1}^{0} + \left(\frac{\tau^4}{144h^4}\right)u_{n-2}^{1} \\
+\left(-\frac{\tau^4}{144h^4}\right)u_{n-2}^{0} - \frac{11}{12}u_n^N + \frac{18}{12}u_n^{N-1} - \frac{9}{12}u_n^{N-2} + \frac{2}{12}u_n^{N-3} = \varphi_n^N, \\
\varphi_n^N = \left(\tau - \frac{\tau^2}{3} + \frac{1}{2}\tau e^{-1}\right)\sin(x_n), 2 \le n \le M-2, \\
u_3^k = 4u_2^k - 5u_1^k, u_{M-3}^k = 4u_{M-2}^k - 5u_{M-1}^k, 0 \le k \le N.
\end{cases} \tag{4.15}
$$

Düzenlemeden sonra, önceki bölümlerde sunduğumuz lineer sistemin aynısı elde edilir.

Burada, $\varphi_n^k = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}, \quad 0 \le k \le N,$

$$\varphi_n^0 = \left(1 - \frac{1}{2}e^{-1}\right)\sin\left(x_n\right), \ \varphi_n^N = \left(\tau - \frac{\tau^2}{3} + \frac{1}{2}\tau e^{-1}\right)\sin\left(x_n\right),$$

$$\varphi_n^k = \left(\tfrac{4}{3}\exp\left(-t_k\right) + \tfrac{1}{3}\left(\exp\left(-t_{k+1}\right) + \exp\left(-t_{k-1}\right)\right)\right)\sin\left(x_n\right), \ 1 \le k \le N-1,$$

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ x & 0 & \cdots & 0 & 0 & 0 \\ 0 & x & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x & 0 \\ a & aa & \cdots & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ ww & w & v & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & ww & w & v & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & ww & w & v \\ b & bb & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{2} \\ m & n & l & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & m & n & l & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & m & n & l \\ c & cc & 0 & 0 & \cdots & -\frac{1}{6} & \frac{3}{4} & -\frac{3}{2} & \frac{12}{11} \end{bmatrix}_{(N+1)\times(N+1)} , D = B,\ E = A,$$

$$R = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{(N+1)\times(N+1)} ,$$

ve

$$U_s^k = \begin{bmatrix} u_s^0 \\ u_s^1 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1} , 0 \le k \le N, s = n-2, n-1, n, n+1, n+2$$

dir. A, B ve C matrislerinin elemanları için

$$x=\frac{\tau^2}{12h^4}, v=-\frac{1}{6h^2}-\frac{\tau^2}{3h^4}, w=-\frac{2}{3h^2}, ww=-\frac{1}{6h^2},\ m=\frac{1}{\tau^2}+\frac{1}{3h^2}+\frac{\tau^2}{2h^4},$$

$$n=-\frac{2}{\tau^2}+\frac{4}{3h^2},\ l=\frac{1}{\tau^2}+\frac{1}{3h^2}, a=-\frac{\tau^2}{144h^4}, aa=\frac{\tau^2}{144h^4}, b=\frac{\tau^4}{36h^4}+\frac{\tau^2}{12h^2},$$

$$bb=-\frac{\tau^4}{36h^4}-\frac{\tau^2}{12h^2},\ c=-1-\frac{\tau^2}{6h^2}-\frac{\tau^4}{24h^4},\ cc=1+\frac{\tau^2}{6h^2}+\frac{\tau^4}{24h^4}$$

gösterimleri kullanılmıştır. Fark şeması (4.14) ün çözümünde kullanılan yöntem bir önceki başlıkta sunulan yöntemle tamamen aynıdır. Böylece, lineer sistemden, formüllerden, matrislerden ve üçüncü mertebeden yaklaşımlı fark şemasından yararlanarak, lokal olmayan sınır değer problemi (4.2) nin yaklaşık çözümü elde edilir. Ayrıca, Matlab programı kullanılarak değişik M ve N değerleri için elde edilen sayısal sonuçlar hata analizi başlığında, yazılmış olan program kodları ise ekler bölümünde sunulmuştur.

4.5 Lokal Olmayan Hiperbolik Tip Sınır Değer Probleminin Yaklaşık Çözümü İçin Zamana Bağlı Dördüncü Mertebeden Doğruluk Fark Şeması

Bu kısımda, yaklaşım formülleri (4.3), (4.4), (4.5), (4.6),

$$\frac{25u(1)-48u(1-\tau)+36u(1-2\tau)-16u(1-3\tau)+3u(1-4\tau)}{12\tau}-u'(1)=O(\tau^4)$$

ve fark şeması (3.37) kullanılarak, lokal olmayan sınır değer problemi (4.2) nin zamana yani t ye bağlı üçüncü mertebeden, uzay boyuta yani x e bağlı ikinci mertebeden yaklaşık çözümü için fark şeması (4.16) elde edildi.

$$
\begin{cases}
\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\frac{5}{6}\left(\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}\right) \\
-\frac{1}{12}\left(\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\right) \\
-\frac{\tau^2}{72}\left(\frac{u_{n+2}^k-4u_{n+1}^k+6u_n^k-4u_{n-1}^k+u_{n-2}^k}{h^4}\right) \\
+\frac{\tau^2}{144}\left(\frac{u_{n+2}^{k+1}-4u_{n+1}^{k+1}+6u_n^{k+1}-4u_{n-1}^{k+1}+u_{n-2}^{k+1}}{h^4}\right. \\
\left.+\frac{u_{n+2}^{k-1}-4u_{n+1}^{k-1}+6u_n^{k-1}-4u_{n-1}^{k-1}+u_{n-2}^{k-1}}{h^4}\right)=\varphi_n^k, \\
\varphi_n^k=\left(\frac{5}{3}(\exp(-t_k))+\frac{1}{6}(\exp(-t_{k+1})+\exp(-t_{k-1}))\right)\sin(x_n), \\
t_k=k\tau,\ 1\le k\le N-1, N\tau=1, x_n=nh, 1\le n\le M-1, Mh=\pi, \\
u_n^0-\frac{1}{2}u_n^N=\varphi_n^0, \varphi_n^0=(1-\frac{1}{2}e^{-1})\sin(x_n), 0\le n\le M, \\
u_n^1-u_n^0-\frac{\tau^2}{12}\left(\frac{(u_{n+1}^1-u_{n+1}^0)-2(u_n^1-u_n^0)+(u_{n-1}^1-u_{n-1}^0)}{h^2}\right) \\
+\frac{\tau^4}{144}\left(\frac{u_{n+2}^1-u_{n+2}^0-4\left(u_{n+1}^1-u_{n+1}^0\right)+6\left(u_n^1-u_n^0\right)-4\left(u_{n-1}^1-u_{n-1}^0\right)+u_{n-2}^1-u_{n-2}^0}{h^4}\right) \\
+\frac{1}{2}\left(\frac{25u_n^N-48u_n^{N-1}+36u_n^{N-2}-16u_n^{N-3}+3u_n^{N-4}}{12}\right)=\varphi_n^N, 2\le n\le M-2, \\
\varphi_n^N=-\frac{\tau^2}{2}A\sin(x_n)+\tau\left(I-\frac{\tau^2}{12}A\right)(-\sin(x_n))+\tau f_{2,2}+\frac{1}{2}\tau e^{-1}\sin(x_n) \\
f_{2,2}=\left(I-\frac{\tau^2 A}{12}\right)f(0)\tau-\left(\left(I-\frac{5\tau^2 A}{12}\right)-\tau f'(0)\right)\frac{\tau}{2} \\
+(-A\tau f(0)-2f'(0)+\tau f''(0))\frac{\tau^2}{6}+(Af(0)-3f''(0))\frac{\tau^3}{24}, \\
u_0^k=u_M^k=0, 0\le k\le N, \\
u_3^k=4u_2^k-5u_1^k, u_{M-3}^k=4u_{M-2}^k-5u_{M-1}^k, 0\le k\le N.
\end{cases} \tag{4.16}
$$

Problem (4.16), $(N+1)\times(N+1)$ boyutlu lineer denklem sistemi olarak yazılabilir. Sistem yeniden düzenlenerek matris formda yazılacak olursa fark şeması (4.17) elde edilir.

$$
\begin{cases}
\left(\frac{\tau^2}{144h^4}\right)u_{n+2}^{k+1} + \left(-\frac{\tau^2}{72h^4}\right)u_{n+2}^{k} + \left(\frac{\tau^2}{144h^4}\right)u_{n+2}^{k-1} \\
+\left(-\frac{1}{12h^2} - \frac{4\tau^2}{144h^4}\right)u_{n+1}^{k+1} + \left(-\frac{5}{6h^2} + \frac{4\tau^2}{72h^4}\right)u_{n+1}^{k} \\
+\left(-\frac{1}{12h^2} - \frac{4\tau^2}{144h^4}\right)u_{n+1}^{k-1} + \left(\frac{1}{\tau^2} + \frac{1}{6h^2} + \frac{\tau^2}{24h^4}\right)u_{n}^{k+1} \\
+\left(-\frac{2}{\tau^2} + \frac{5}{3h^2} - \frac{\tau^2}{12h^4}\right)u_{n}^{k} + \left(\frac{1}{\tau^2} + \frac{1}{6h^2} + \frac{\tau^2}{12h^4}\right)u_{n}^{k-1} \\
+\left(-\frac{1}{12h^2} - \frac{4\tau^2}{144h^4}\right)u_{n-1}^{k+1} + \left(-\frac{5}{6h^2} + \frac{4\tau^2}{72h^4}\right)u_{n-1}^{k} \\
+\left(-\frac{1}{12h^2} - \frac{4\tau^2}{144h^4}\right)u_{n-1}^{k-1} + \left(\frac{\tau^2}{144h^4}\right)u_{n-2}^{k+1} + \left(-\frac{\tau^2}{72h^4}\right)u_{n-2}^{k} \\
+\left(\frac{\tau^2}{144h^4}\right)u_{n-2}^{k-1} = \varphi_n^k, 1 \le k \le N-1, 2 \le n \le M-2, \\
u_n^0 = \varphi_n^0 = \sin(x_n), 0 \le n \le M, \\
\left(\frac{\tau^4}{144h^4}\right)u_{n+2}^{1} + \left(-\frac{\tau^4}{144h^4}\right)u_{n+2}^{0} + \left(-\frac{\tau^4}{36h^4} - \frac{\tau^2}{12h^2}\right)u_{n+1}^{1} \\
+\left(\frac{\tau^4}{36h^4} + \frac{\tau^2}{12h^2}\right)u_{n+1}^{0} + \left(1 + \frac{\tau^2}{6h^2} + \frac{\tau^4}{24h^4}\right)u_{n}^{1} \\
+\left(-1 - \frac{\tau^2}{6h^2} - \frac{\tau^4}{24h^4}\right)u_{n}^{0} + \left(-\frac{\tau^4}{36h^4} - \frac{\tau^2}{12h^2}\right)u_{n-1}^{1} \\
+\left(\frac{\tau^4}{36h^4} + \frac{\tau^2}{12h^2}\right)u_{n-1}^{0} + \left(\frac{\tau^4}{144h^4}\right)u_{n-2}^{1} + \left(-\frac{\tau^4}{144h^4}\right)u_{n-2}^{0} + \frac{25}{24}u_n^N \\
-\frac{48}{24}u_n^{N-1} + \frac{36}{24}u_n^{N-2} - \frac{16}{24}u_n^{N-3} + \frac{3}{24}u_n^{N-4} = \varphi_n^N, 2 \le n \le M-2, \\
\varphi_n^N = \left(-\tau + \frac{\tau^2}{2} - \frac{\tau^3}{4} + \frac{\tau^4}{12} - \frac{\tau^5}{36} + \frac{\tau^6}{16} - \frac{\tau^7}{108} - \frac{\tau^8}{864} + \frac{1}{2}\tau e^{-1}\right)\sin(x_n), \\
u_0^k = u_M^k = 0, 0 \le k \le N, \\
u_3^k = 4u_2^k - 5u_1^k, u_{M-3}^k = 4u_{M-2}^k - 5u_{M-1}^k, 0 \le k \le N.
\end{cases}
\tag{4.17}
$$

Düzenlemeden sonra önceki bölümlerde sunulan lineer sistemin aynısı elde edilir.

Burada, $\varphi_n^k = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}$, $0 \le k \le N$, $\varphi_n^0 = \left(1 - \frac{1}{2}e^{-1}\right)\sin\left(x_n\right)$,

$$
\varphi_n^N = \left(-\tau + \frac{\tau^2}{2} - \frac{\tau^3}{4} + \frac{\tau^4}{12} - \frac{\tau^5}{36} + \frac{\tau^6}{16} - \frac{\tau^7}{108} - \frac{\tau^8}{864} + \frac{1}{2}\tau e^{-1}\right)\sin\left(x_n\right),
$$

$$
\varphi_n^k = \left(\tfrac{5}{3}\left(\exp\left(-t_k\right)\right) + \tfrac{1}{6}\left(\exp\left(-t_{k+1}\right) + \exp\left(-t_{k-1}\right)\right)\right)\sin\left(x_n\right)\sin\left(x_n\right),\ 1 \le k \le N-1,
$$

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ x & y & x & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & x & y & x & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & x & y & x \\ a & aa & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ v & w & v & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & v & w & v & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & v & w & v \\ b & bb & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ m & n & m & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & m & n & m & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & m & n & m \\ c & cc & 0 & 0 & \cdots & \frac{1}{8} & -\frac{2}{3} & \frac{3}{2} & -2 & \frac{25}{24} \end{bmatrix}_{(N+1)\times(N+1)}, D=B,\ E=A,$$

$$R=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

ve

$$U_s^k=\begin{bmatrix} u_s^0 \\ u_s^1 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1}, 0\le k\le N, s=n-2, n-1, n, n+1, n+2$$

dir. A, B ve C matrislerinin elemanları için

$$x=\frac{\tau^2}{144h^4},\ y=-\frac{\tau^2}{72h^4} v=-\frac{1}{12h^2}-\frac{4\tau^2}{144h^4},\ w=-\frac{5}{6h^2}+\frac{4\tau^2}{72h^4},$$

$$m=\frac{1}{\tau^2}+\frac{1}{6h^2}+\frac{\tau^2}{24h^4},\ n=-\frac{2}{\tau^2}+\frac{5}{3h^2}-\frac{\tau^2}{12h^4},$$

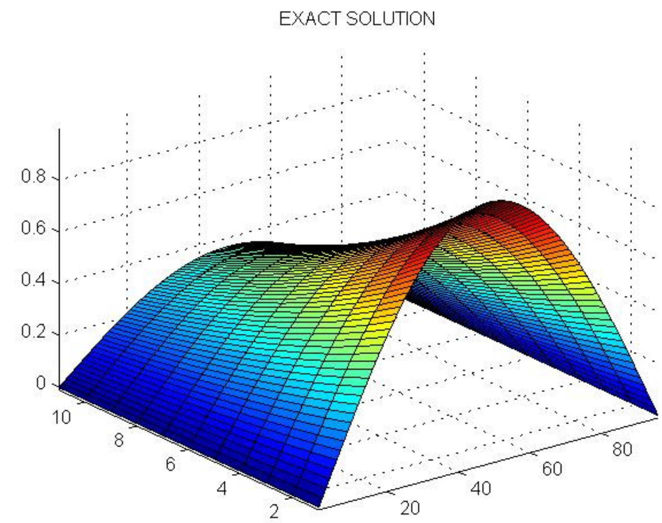
$$a=-\frac{\tau^2}{144h^4},\ aa=\frac{\tau^2}{144h^4},\ b=\frac{\tau^4}{36h^4}+\frac{\tau^2}{12h^2},\ bb=-\frac{\tau^4}{36h^4}-\frac{\tau^2}{12h^2},$$

$$c=-1-\frac{\tau^2}{6h^2}-\frac{\tau^4}{24h^4}, cc=1+\frac{\tau^2}{6h^2}+\frac{\tau^4}{24h^4}$$

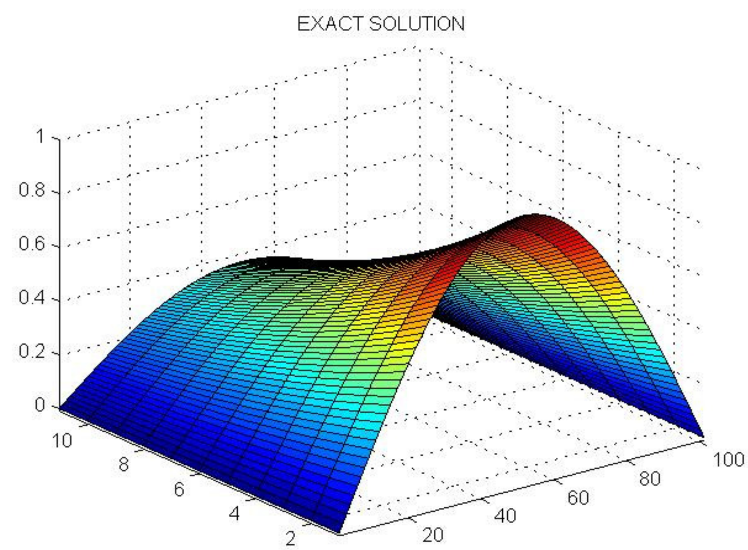
gösterimleri kullanılmıştır. Fark şeması (4.16) nın çözümünde kullanılan yöntem bir önceki başlıkta sunulan yöntemle tamamen aynıdır. Böylece, lineer sistemden, formüllerden, matrislerden ve üçüncü mertebeden yaklaşımlı fark şemasından yararlanarak, lokal olmayan sınır değer problemi (4.2) nin yaklaşık çözümü elde edilir. Ayrıca, Matlab programı kullanılarak değişik M ve N değerleri için elde edilen sayısal sonuçlar hata analizi başlığında, yazılmış olan program kodları ise ekler bölümünde sunulmuştur.

4.6 Hata Analizi

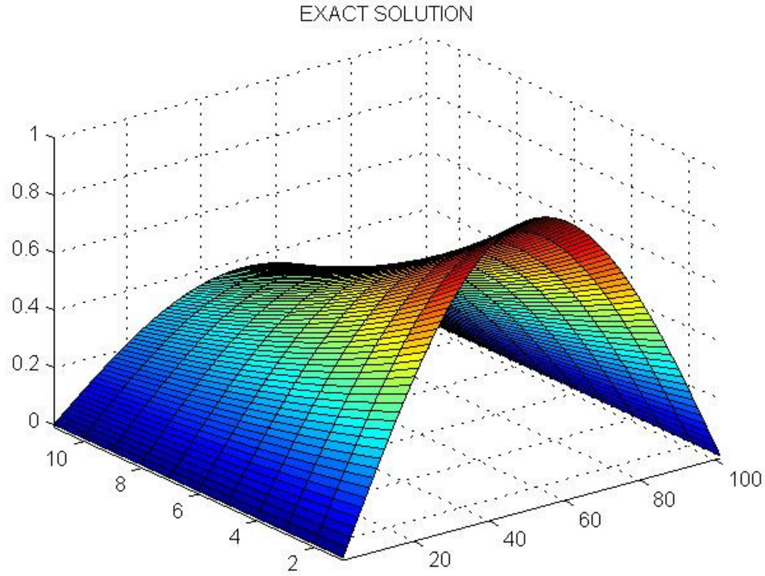
Bu kısımda, hiperbolik tip Cauchy problemi (4.1) in ve hiperbolik tip lokal olmayan sınır değer problemi (4.2) nin yaklaşık çözümleri için üçüncü ve dördüncü mertebeden fark şemaları kullanılarak elde edilen grafiklerin bir bölümü sırasıyla sunuldu.



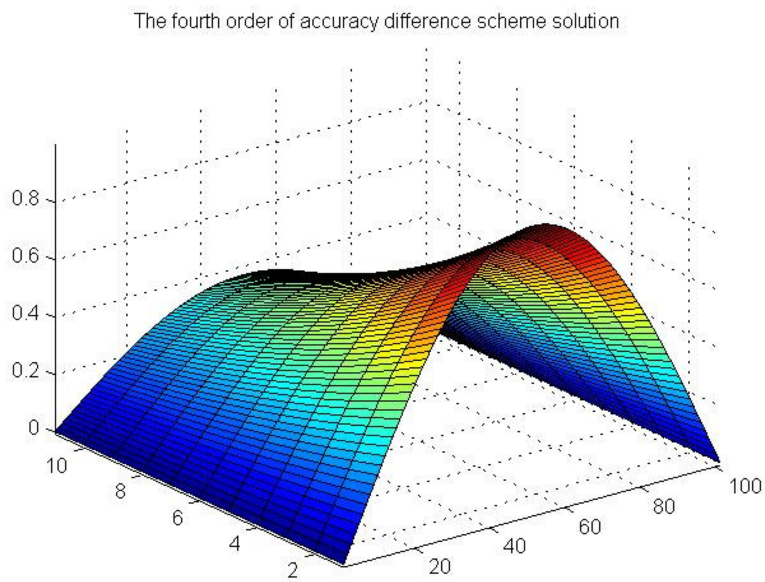
Şekil 4.1. Cauchy problemi (4.1) in üçüncü mertebeden kesin çözümünün grafiği



Şekil 4.2. Cauchy problemi (4.1) in üçüncü mertebeden yaklaşık çözümün grafiği



Şekil 4.3. Cauchy problemi (4.1) in dördüncü mertebeden kesin çözümünün grafiği



Şekil 4.4. Cauchy problemi (4.1) in dördüncü mertebeden yaklaşık çözümünün grafiği

Lokal olmayan hiperbolik tip sınır değer problemi (4.2) nin kesin çözümü de Cauchy promlemi (4.1) ile aynı olduğundan benzer grafikler elde edilir. Bu sebeple birbirlerine çok benzer grafiklerin daha fazla yer tutmaması için teze koymaya gerek duyulmamıştır.

Bütün (t_k, x_n) noktalarında, $u(t_k, x_n)$ kesin çözümü, U_n^k yaklaşık çözümü göstermek üzere hata kıyaslaması

$$E_M^N = \max_{1 \le k \le N-1} \left(\sum_{n=1}^{M-1} |u(t_k, x_n) - U_n^k|^2 h \right)^{\frac{1}{2}}$$

formülü ile hesaplandı. Sonuçlar aşağıdaki tablolarda sunuldu.

Problem (4.1) in yaklaşık çözümleri		
Üçüncü mertebeden çözüm	$E_{N=10}^{M=30}$: 0,7107.10^{-4}	$E_{N=20}^{M=50}$: 0,7645.10^{-5}
Dördüncü mertebeden çözüm	$E_{N=10}^{M=100}$: 0,2459.10^{-4}	$E_{N=20}^{M=400}$: 0,1628.10^{-5}

Çizelge 4.1. Problem (4.1) in yaklaşık çözümleri için hata analizi

Problem(4.2) nin yaklaşık çözümleri		
Üçüncü mertebeden çözüm	$E_{N=10}^{M=30}$: 0,1226.10^{-3}	$E_{N=20}^{M=50}$: 0,1484.10^{-4}
Dördüncü mertebeden çözüm	$E_{N=10}^{M=100}$: 0,1488.10^{-4}	$E_{N=20}^{M=400}$: 0,1066.10^{-5}

Çizelge 4.2. Problem (4.2) nin yaklaşık çözümleri için hata analizi

Görüldüğü üzere dördüncü mertebeden fark şemalarının çözümleri ile gerçek çözüm arasındaki hata, üçüncü mertebeden fark şemalarında elde edilenlere kıyasla daha küçüktür. Sonuç olarak dördüncü mertebeden fark şemaları, üçüncü mertebeden fark şemalarına göre daha iyi sonuçlar vermektedir.

5. SONUÇLAR

Bu tezde hiperbolik tip Cauchy ve hiperbolik tip lokal olmayan sınır değer problemlerinin çözümlerinin kararlılığı konuları ele alındı ve aşağıdaki orijinal sonuçlar elde edildi.

Hilbert uzayında kendine eşlenik, pozitif tanımlı A operatörü kullanılan soyut (abstract) hiperbolik Cauchy problemi

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = f(t) \quad (0 \le t \le 1), \\ u(0) = \varphi, u'(0) = \psi \end{cases} \tag{5.1}$$

in çözümü için;

• A operatörünün tam kuvvetlerinden oluşan üçüncü mertebeden kararlı doğruluk fark şeması

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{2}{3}Au_k + \frac{1}{6}A(u_{k+1} + u_{k-1}) \\ +\frac{1}{12}\tau^2 A^2 u_{k+1} = f_k, f_k = \frac{2}{3}f(t_k) + \frac{1}{6}(f(t_{k+1}) + f(t_{k-1})) \\ -\frac{1}{12}\tau^2(-Af(t_{k+1}) + f''(t_{k+1})), 1 \le k \le N-1, \\ u_0 = \varphi, \left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)\tau^{-1}(u_1 - u_0) \\ = -\frac{\tau}{2}A\varphi + \left(I - \frac{\tau^2}{12}A\right)\psi + f_{1,1}\tau, \\ f_{1,1}(x) = \frac{1}{2}f(0) + \frac{\tau}{6}f'(0) \end{cases}$$

ve fark şemasının çözümü için kararlılık kestirimleri elde edildi,

• A operatörünün tamsayı kuvvetlerinden oluşan dördüncü mertebeden kararlı doğruluk fark şeması

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{5}{6}Au_k+\frac{1}{12}A(u_{k+1}+u_{k-1}) \\ -\frac{\tau^2}{72}A^2u_k+\frac{\tau^2}{144}A^2(u_{k+1}+u_{k-1})=f_k, \\ f_k=\frac{5}{6}f(t_k)+\frac{1}{12}(f(t_{k+1})+f(t_{k-1}))+\frac{\tau^2}{72}(-Af(t_k)+f''(t_k)) \\ -\frac{1}{144}\tau^2(-A(f(t_{k+1})+f(t_{k-1}))+f''(t_{k+1})+f''(t_{k-1})), \\ 1\le k\le N-1, t_k=k\tau, N\tau=1, \\ u_0=\varphi, \left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\tau^{-1}(u_1-u_0) \\ =-\frac{\tau}{2}A\varphi+\left(I-\frac{\tau^2A}{12}\right)\psi+f_{2,2}\tau, \\ f_{2,2}(x)=\frac{1}{2}f(0)+\frac{\tau}{6}f'(0)+\frac{\tau^2}{24}f''(0) \end{cases}$$

ve fark şemasının çözümü için kararlılık kestirimleri elde edildi.

Hilbert uzayında kendine eşlenik pozitif tanımlı A operatörü kullanılan abstract hiperbolik lokal olmayan sınır değer problemi

$$\begin{cases} \frac{d^2u(t)}{dt^2}+Au(t)=f(t) \quad (0\le t\le 1), \\ u(0)=\alpha u(1)+\varphi,\ u'(0)=\beta u'(1)+\psi \end{cases} \tag{5.2}$$

nin çözümü için;

• A operatörünün tamsayı kuvvetlerinden oluşan üçüncü mertebeden kararlı doğruluk fark şeması

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{2}{3}Au_k+\frac{1}{6}A(u_{k+1}+u_{k-1}) \\ +\frac{1}{12}\tau^2A^2u_{k+1}=f_k, f_k=\frac{2}{3}f(t_k)+\frac{1}{6}(f(t_{k+1})+f(t_{k-1})) \\ -\frac{1}{12}\tau^2(-Af(t_{k+1})+f''(t_{k+1})), t_k=k\tau, 1\le k\le N-1, N\tau=1, \\ u_0=\alpha u_N+\varphi, \left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\tau^{-1}(u_1-u_0)+\frac{\tau}{2}Au_0-\tau f_{1,1} \\ =\beta\left(I-\frac{\tau^2A}{12}\right)\left(\dfrac{7u_N-8u_{N-1}+u_{N-2}}{6\tau}+\frac{\tau}{3}(f_N-Au_N)\right)+\left(I-\frac{\tau^2A}{12}\right)\psi, \\ f_{1,1}(x)=\frac{1}{2}f(0)+\frac{\tau}{6}f'(0) \end{cases}$$

ve fark şemasının çözümü için kararlılık kestirimleri elde edildi.

• A operatörünün tamsayı kuvvetlerinden oluşan dördüncü mertebeden kararlı doğruluk fark şeması

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{5}{6}Au_k + \frac{1}{12}A(u_{k+1} + u_{k-1}) \\ -\frac{\tau^2}{72}A^2u_k + \frac{\tau^2}{144}A^2(u_{k+1} + u_{k-1}) = f_k, \\ f_k = \frac{5}{6}f(t_k) + \frac{1}{12}(f(t_{k+1}) + f(t_{k-1})) + \frac{\tau^2}{72}(-Af(t_k) + f''(t_k)) \\ -\frac{1}{144}\tau^2(-A(f(t_{k+1}) + f(t_{k-1})) + f''(t_{k+1}) + f''(t_{k-1})), \\ t_k = k\tau, 1 \le k \le N-1, N\tau = 1, \\ u_0 = \alpha u_N + \varphi, \left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)\tau^{-1}(u_1 - u_0) + \frac{\tau}{2}Au_0 - \tau f_{2,2} \\ = \beta\left(I - \frac{\tau^2 A}{12}\right)\left(\frac{85u_N - 108u_{N-1} + 27u_{N-2} - 4u_{N-3}}{66\tau} + \frac{3\tau}{11}(f_N - Au_N)\right) \\ +\left(I - \frac{\tau^2 A}{12}\right)\psi, \\ f_{2,2}(x) = \frac{1}{2}f(0) + \frac{\tau}{6}f'(0) + \frac{\tau^2}{24}f''(0) \end{cases}$$

ve fark şemasının çözümü için kararlılık kestirimleri elde edildi.

• Teorik sonuçlar nümerik deneylerle desteklendi.

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EKLER

EK 1. Hiperbolik tip Cauchy probleminin üçüncü mertebeden nümerik çözümü için Matlab Programı kodu

```
function thirdordercauchy(N,M)
close; close; close;
 N=20; M=30;
tau=1/N; h=pi/M;

x= (tau^2)/(12*(h^4));

vvv= -1/(6*(h^2))-(tau^2)/(3*(h^4));
vv= -2/(3*(h^2));
v= -1/(6*(h^2));



m= 1/(tau^2)+1/(3*(h^2));
mm=-2/(tau^2)+4/(3*(h^2));
mmm=1/(tau^2)+1/(3*(h^2))+(tau^2)/(2*(h^4));

a= (-tau^4/(144*h^4));
aa= tau^4/(144*h^4);
b=((tau^2)/(12*(h^2))+(tau^4)/(36*(h^4)));
bb= (-(tau^2)/(12*(h^2))-(tau^4)/(36*(h^4)));
c=(-1-(tau^2)/(6*(h^2))-(tau^4)/(24*(h^4)));
cc=(1+(tau^2)/(6*(h^2))+(tau^4)/(24*(h^4)));

    A(1,1)=0;
    for i=1:N-1 ; A(i+1,i+2)= x ; end;
    A(N+1,1)=a; A(N+1,2)=aa;A;
    A;
    B(1,1) =0 ;
    for i=1:N-1 ; B(i+1,i)=v ; end;
    for i=1:N-1 ; B(i+1,i+1)= vv ; end;
    for i=1:N-1 ; B(i+1,i+2)=vvv; end;
    B(N+1,1)=b; B(N+1,2)=bb; B;

 C(1,1)=1;
    for i=1:N-1 ; C(i+1,i)= m ; end;
    for i=1:N-1 ; C(i+1,i+1)= mm ; end;
    for i=1:N-1 ; C(i+1,i+2)= mmm ; end;
  C(N+1,1)=c; C(N+1,2)=cc;
     C;
  D=B;
   E=A;
    for i=1:N+1; R(i,i)=1 ; end;
    R;
   'fii(j) finding ';
    for k=1:N-1;
     for j=1:M+1;
         fii(1,j:j) =sin((j)*h) ;
```

```
     fii(k+1,j:j) =((5/3-(tau^2/36)*(1-(k*tau)^2))*(exp(-(k)*tau)*
sin((j)*h))+...
         (1/6+(tau^2/72)*(1-((k+1)*tau)^2))*(exp(-(k+1)*tau)*
sin((j)*h))+...
         (1/6+(tau^2/72)*(1-((k-1)*tau)^2))*(exp(-(k-1)*tau)*
sin((j)*h)));
        fii(N+1,j:j)=(-tau+(1/2)*tau^2-(1/4)*tau^3+(1/4)*tau^4-
(7/36)*tau^5-(1/144)*tau^6+(1/216)*tau^7)*...
            sin((j)*h);
   end;
    end;fii ;

    alpha(1:N+1,1:N+1,1:1) = 0*eye(N+1) ;
    betha(1:N+1,1:N+1,1:1) = 0*eye(N+1) ;
    gamma(N+1,1:1)= 0 ;
    alpha(1:N+1,1:N+1,2:2) = (4/5)*eye(N+1) ;
    betha(1:N+1,1:N+1,2:2) = (-1/5)*eye(N+1);
    gamma(N+1,2:2)= 0 ;


for n = 2:M-2 ;
bebek = C + D*alpha(:,: , n:n ) + E*betha(:,:,n-1 : n-1)+
E*alpha(:,:,n-1:n-1)*alpha(:,:, n:n) ;
betha(:,:,n+1:n+1 ) = -inv( bebek )*(A) ;
alpha(:,:,n+1:n+1) = -inv(bebek )*(B +D*betha(:,:,n:n)+ E *
alpha(:,:,n-1:n-1)* betha(:,:,n) ) ;
gamma(:,n+1:n+1) = inv( bebek )*(R*fii(:,n:n) - D * gamma(:,n:n)-E *
alpha(:,:,n-1:n-1)* gamma(:,n:n) - E*gamma(:, n-1 : n-1) ) ;
end;
U(1:N+1,1:N+1)=nan;
U( 1:N+1, M:M ) = 0 ;
U( :, M-1:M-1 ) = inv( (betha(:,:,M-2:M-2) + 5*eye(N+1))- (4*eye(N+1)-
alpha(:,:,M-2:M-2) )*alpha(:,:,M-1:M-1))*((4*eye(N+1)-alpha(:,:,M-2:M-
2))*gamma(: , M-1:M-1)- gamma(: , M-2:M-2) );
U(: , M-2:M-2 ) =inv(4*eye(N+1)-alpha(:,:,M-2:M-2))*((betha(:,:,M-2:M-
2)+5*eye(N+1))*U(:,M-1:M-1)+gamma(:,M-2:M-2));

for z = M-3:-1:1;
U(:,z:z
)=alpha(:,:,z+1:z+1)*U(:,z+1:z+1)+betha(:,:,z+1:z+1)*U(:,z+2:z+2)+gamm
a(:,z+1:z+1);end;

for z = 1 : M ;
p(:,z+1:z+1)=U(:,z:z);
end;

 'EXACT SOLUTION OF THIS PROBLEM' ;
    for d=1:N+1;
        for f=1:M+1 ;
    es( d, f )=exp(-(d-1)*tau)* sin((f-1)*h);
    end;
    end;es;
       figure ;
m(1,1)=min(min(es))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(es) ; rotate3d ;axis tight;
```

```
title('The exact solution');
figure ;
m(1,1)=min(min(p))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(p) ; rotate3d ;axis tight;
title ('The third order of accuracy difference scheme solution');
% .ERROR ANALYSIS.;
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)));
relativeerror=max(max((abs(es-p))))/max(max(abs(p)) );
cevap = [maxerror,relativeerror]
```

EK 2. Hiperbolik tip Cauchy probleminin dördüncü mertebeden nümerik çözümü için Matlab Programı kodu

```
function fourthordercauchy(N,M)
close; close; close;
 N=10; M=100;
tau=1/N; h=pi/M;

x= tau^2/(144*h^4);
y=(-(tau^2)/(72*(h^4)));
v= (-(1/(12*(h^2))+(tau^2)/(36*(h^4))));
w= (-5/(6*(h^2))+ (tau^2)/(18*(h^4)));
m= (1/(tau^2)+1/(6*(h^2))+(tau^2)/(24*(h^4)));
n= (-2/(tau^2)+5/(3*(h^2))-(tau^2)/(12*(h^4)));
a= (-tau^4/(144*h^4));
aa= tau^4/(144*h^4);
b=((tau^2)/(12*(h^2))+(tau^4)/(36*(h^4)));
bb= (-(tau^2)/(12*(h^2))-(tau^4)/(36*(h^4)));
c=(-1-(tau^2)/(6*(h^2))-(tau^4)/(24*(h^4)));
cc=(1+(tau^2)/(6*(h^2))+(tau^4)/(24*(h^4)));

    A(1,1)=0;
    for i=1:N-1 ; A(i+1,i)= x ; end;
    for i=1:N-1 ; A(i+1,i+1)= y ; end;
    for i=1:N-1 ;A(i+1,i+2)= x ; end; A;
    A(N+1,1)=a; A(N+1,2)=aa;
    A;
    B(1,1) =0 ;
    for i=1:N-1 ; B(i+1,i)=v ; end;
    for i=1:N-1 ; B(i+1,i+1)= w ; end;
    for i=1:N-1 ; B(i+1,i+2)= v ; end;
    B(N+1,1)=b; B(N+1,2)=bb; B;

 C(1,1)=1;
    for i=1:N-1 ; C(i+1,i)= m ; end;
    for i=1:N-1 ; C(i+1,i+1)= n ; end;
    for i=1:N-1 ; C(i+1,i+2)= m ; end; C;
  C(N+1,1)=c; C(N+1,2)=cc;
     C;
  D=B;
   E=A;
    for i=1:N+1; R(i,i)=1 ; end;
    R;
  'fii(j) finding ';
    for k=1:N-1;
     for j=1:M+1;
         fii(1,j:j) =sin((j)*h) ;
     fii(k+1,j:j) =(5/3*(exp(-(k)*tau))+(1/6)*((exp(-(k+1)*tau))+exp(-
(k-1)*tau)))* sin((j)*h);

   fii(N+1,j:j)=(-tau+(1/2)*tau^2-(1/4)*tau^3+(1/12)*tau^4-
(1/36)*tau^5+(1/16)*tau^6-(1/108)*tau^7-(1/864)*tau^8)...
            *sin((j)*h);
   end;
    end;fii ;
```

```
    alpha(1:N+1,1:N+1,1:1) = 0*eye(N+1) ;
    betha(1:N+1,1:N+1,1:1) = 0*eye(N+1) ;
    gamma(N+1,1:1)= 0 ;
    alpha(1:N+1,1:N+1,2:2) = (4/5)*eye(N+1) ;
    betha(1:N+1,1:N+1,2:2) = (-1/5)*eye(N+1);
    gamma(N+1,2:2)= 0 ;


for n = 2:M-2 ;
bebek = C + D*alpha(:,: , n:n ) + E*betha(:,:,n-1 : n-1)+
E*alpha(:,:,n-1:n-1)*alpha(:,:, n:n) ;
betha(:,:,n+1:n+1 ) = -inv( bebek )*(A) ;
alpha(:,:,n+1:n+1) = -inv(bebek )*(B +D*betha(:,:,n:n)+ E *
alpha(:,:,n-1:n-1)* betha(:,:,n) ) ;
gamma(:,n+1:n+1) = inv( bebek )*(R*fii(:,n:n) - D * gamma(:,n:n)-E *
alpha(:,:,n-1:n-1)* gamma(:,n:n) - E*gamma(:, n-1 : n-1) ) ;
end;
U(1:N+1,1:N+1)=nan;
U( 1:N+1, M:M ) = 0 ;
U( :, M-1:M-1 ) = inv( (betha(:,:,M-2:M-2) + 5*eye(N+1))- (4*eye(N+1)-
alpha(:,:,M-2:M-2) )*alpha(:,:,M-1:M-1))*((4*eye(N+1)-alpha(:,:,M-2:M-
2))*gamma(: , M-1:M-1)- gamma(: , M-2:M-2) );
U(: , M-2:M-2 ) =inv(4*eye(N+1)-alpha(:,:,M-2:M-2))*((betha(:,:,M-2:M-
2)+5*eye(N+1))*U(:,M-1:M-1)+gamma(:,M-2:M-2));

for z = M-3:-1:1;
U(:,z:z
)=alpha(:,:,z+1:z+1)*U(:,z+1:z+1)+betha(:,:,z+1:z+1)*U(:,z+2:z+2)+gamm
a(:,z+1:z+1);end;

for z = 1 : M ;
p(:,z+1:z+1)=U(:,z:z);
end;

 'EXACT SOLUTION OF THIS PROBLEM' ;
    for d=1:N+1;
        for f=1:M+1 ;
    es( d, f )=exp(-(d-1)*tau)* sin((f-1)*h);
    end;
    end;es;


    figure ;
m(1,1)=min(min(p))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(es) ; rotate3d ;axis tight;
title('EXACT SOLUTION');
figure ;
m(1,1)=min(min(p))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(p) ; rotate3d ;axis tight;
title ('The fourth order of accuracy difference scheme solution');
```

```
% .ERROR ANALYSIS.;
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)));
relativeerror=max(max((abs(es-p))))/max(max(abs(p)) );
cevap = [maxerror,relativeerror]
```

EK 3. Lokal olmayan hiperbolik tip sınır değer probleminin üçüncü mertebeden nümerik çözümü için Matlab Programı kodu

```
function thirdordeNL(N,M)
close; close; close;
 N=20; M=30;
tau=1/N; h=pi/M;
x= (tau^2)/(12*(h^4));

vvv= -1/(6*(h^2))-(tau^2)/(3*(h^4));
vv= -2/(3*(h^2));
v= -1/(6*(h^2));


m= 1/(tau^2)+1/(3*(h^2));
mm=-2/(tau^2)+4/(3*(h^2));
mmm=1/(tau^2)+1/(3*(h^2))+(tau^2)/(2*(h^4));

a= (-tau^4/(144*h^4));
aa= tau^4/(144*h^4);
b=((tau^2)/(12*(h^2))+(tau^4)/(36*(h^4)));
bb= (-(tau^2)/(12*(h^2))-(tau^4)/(36*(h^4)));
c=(-1-(tau^2)/(6*(h^2))-(tau^4)/(24*(h^4)));
cc=(1+(tau^2)/(6*(h^2))+(tau^4)/(24*(h^4)));

   A(1,1)=0;
    for i=1:N-1 ; A(i+1,i+2)= x ; end;
    A(N+1,1)=a; A(N+1,2)=aa;A;
    A;
    B(1,1) =0 ;
    for i=1:N-1 ; B(i+1,i)=v ; end;
    for i=1:N-1 ; B(i+1,i+1)= vv ; end;
    for i=1:N-1 ; B(i+1,i+2)=vvv; end;
    B(N+1,1)=b; B(N+1,2)=bb; B;

C(1,1)=1;C(1,N+1)=-1/2;
    for i=1:N-1 ; C(i+1,i)= m ; end;
    for i=1:N-1 ; C(i+1,i+1)= mm ; end;
    for i=1:N-1 ; C(i+1,i+2)= mmm ; end;
  C(N+1,1)=c; C(N+1,2)=cc; C(N+1,N+1)=11/(2*6); C(N+1,N)=-18/(2*6);
  C(N+1,N-1)=9/(2*6); C(N+1,N-2)=-2/(2*6);
     C;
  D=B;
   E=A;
    for i=1:N+1; R(i,i)=1 ; end;
    R;
   'fii(j) finding ';
    for k=1:N-1;
     for j=1:M+1;
         fii(1,j:j) =(1-(1/2)*exp(-1))*sin((j)*h) ;
     fii(k+1,j:j) =((5/3-(tau^2/36)*(1-(k*tau)^2))*(exp(-(k)*tau)*
sin((j)*h))+...
         (1/6+(tau^2/72)*(1-((k+1)*tau)^2))*(exp(-(k+1)*tau)*
sin((j)*h))+...
```

```
         (1/6+(tau^2/72)*(1-((k-1)*tau)^2))*(exp(-(k-1)*tau)*
sin((j)*h)));
        fii(N+1,j:j)=((-tau+(1/2)*tau^2-(1/4)*tau^3+(1/4)*tau^4-
(7/36)*tau^5-(1/144)*tau^6+(1/216)*tau^7))*sin((j)*h)+...
            (-tau*(1/2)*exp(-1))*sin((j)*h);
   end;
    end;fii ;

    alpha(1:N+1,1:N+1,1:1) = 0*eye(N+1) ;
    betha(1:N+1,1:N+1,1:1) = 0*eye(N+1) ;
    gamma(N+1,1:1)= 0 ;
    alpha(1:N+1,1:N+1,2:2) = (4/5)*eye(N+1) ;
    betha(1:N+1,1:N+1,2:2) = (-1/5)*eye(N+1);
    gamma(N+1,2:2)= 0 ;


for n = 2:M-2 ;
bebek = C + D*alpha(:,: , n:n ) + E*betha(:,:,n-1 : n-1)+
E*alpha(:,:,n-1:n-1)*alpha(:,:, n:n) ;
betha(:,:,n+1:n+1 ) = -inv( bebek )*(A) ;
alpha(:,:,n+1:n+1) = -inv(bebek )*(B +D*betha(:,:,n:n)+ E *
alpha(:,:,n-1:n-1)* betha(:,:,n) ) ;
gamma(:,n+1:n+1) = inv( bebek )*(R*fii(:,n:n) - D * gamma(:,n:n)-E *
alpha(:,:,n-1:n-1)* gamma(:,n:n) - E*gamma(:, n-1 : n-1) ) ;
end;
U(1:N+1,1:N+1)=nan;
U( 1:N+1, M:M ) = 0 ;
U( :, M-1:M-1 ) = inv( (betha(:,:,M-2:M-2) + 5*eye(N+1))- (4*eye(N+1)-
alpha(:,:,M-2:M-2) )*alpha(:,:,M-1:M-1))*((4*eye(N+1)-alpha(:,:,M-2:M-
2))*gamma(: , M-1:M-1)- gamma(: , M-2:M-2) );
U(: , M-2:M-2 ) =inv(4*eye(N+1)-alpha(:,:,M-2:M-2))*((betha(:,:,M-2:M-
2)+5*eye(N+1))*U(:,M-1:M-1)+gamma(:,M-2:M-2));

for z = M-3:-1:1;
U(:,z:z
)=alpha(:,:,z+1:z+1)*U(:,z+1:z+1)+betha(:,:,z+1:z+1)*U(:,z+2:z+2)+gamm
a(:,z+1:z+1);end;

for z = 1 : M ;
p(:,z+1:z+1)=U(:,z:z);
end;

 'EXACT SOLUTION OF THIS PROBLEM' ;
    for d=1:N+1;
        for f=1:M+1 ;
    es( d, f )=exp(-(d-1)*tau)* sin((f-1)*h);
    end;
    end;es;

figure ;
n(1,1)=min(min(es))-0.01;
n(2,2)=nan;
surf(n);
hold;
surf(es) ; rotate3d ;axis tight;
title('The exact solution');
figure ;
```

```
m(1,1)=min(min(p))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(p) ; rotate3d ;axis tight;
title ('The third order of accuracy difference scheme solution');
% .ERROR ANALYSIS.;
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)));
relativeerror=max(max((abs(es-p))))/max(max(abs(p)) );
cevap = [maxerror,relativeerror]
```

EK 4. Lokal olmayan hiperbolik tip sınır değer probleminin dördüncü mertebeden nümerik çözümü için Matlab Programı kodu

```
function fourthorderNL(N,M)
close; close; close;
 N=20; M=400;
tau=1/N; h=pi/M;

x= tau^2/(144*h^4);
y=(-(tau^2)/(72*(h^4)));
v= (-(1/(12*(h^2))+(tau^2)/(36*(h^4))));
w= (-5/(6*(h^2))+ (tau^2)/(18*(h^4)));
m= (1/(tau^2)+1/(6*(h^2))+(tau^2)/(24*(h^4)));
n= (-2/(tau^2)+5/(3*(h^2))-(tau^2)/(12*(h^4)));
a= (-tau^4/(144*h^4));
aa= tau^4/(144*h^4);
b=((tau^2)/(12*(h^2))+(tau^4)/(36*(h^4)));
bb= (-(tau^2)/(12*(h^2))-(tau^4)/(36*(h^4)));
c=(-1-(tau^2)/(6*(h^2))-(tau^4)/(24*(h^4)));
cc=(1+(tau^2)/(6*(h^2))+(tau^4)/(24*(h^4)));

    A(1,1)=0;
    for i=1:N-1 ; A(i+1,i)= x ; end;
    for i=1:N-1 ; A(i+1,i+1)= y ; end;
    for i=1:N-1 ;A(i+1,i+2)= x ; end; A;
    A(N+1,1)=a; A(N+1,2)=aa;
    A;
    B(1,1) =0 ;
    for i=1:N-1 ; B(i+1,i)=v ; end;
    for i=1:N-1 ; B(i+1,i+1)= w ; end;
    for i=1:N-1 ; B(i+1,i+2)= v ; end;
    B(N+1,1)=b; B(N+1,2)=bb; B;

 C(1,1)=1;C(1,N+1)=-1/2;
    for i=1:N-1 ; C(i+1,i)= m ; end;
    for i=1:N-1 ; C(i+1,i+1)= n ; end;
    for i=1:N-1 ; C(i+1,i+2)= m ; end; C;
 C(N+1,1)=c; C(N+1,2)=cc; C(N+1,N+1)=25/(2*12); C(N+1,N)=-48/(2*12);
  C(N+1,N-1)=36/(2*12); C(N+1,N-2)=-16/(2*12);C(N+1,N-3)=3/(2*12);
     C;
  D=B;
   E=A;
    for i=1:N+1; R(i,i)=1 ; end;
    R;
  'fii(j) finding ';
    for k=1:N-1;
     for j=1:M+1;
         fii(1,j:j) =(1-(1/2)*exp(-1))*sin((j)*h) ;
     fii(k+1,j:j) =(5/3*(exp(-(k)*tau))+(1/6)*((exp(-(k+1)*tau))+exp(-
(k-1)*tau)))* sin((j)*h);
   fii(N+1,j:j)=(-tau+(1/2)*tau^2-(1/4)*tau^3+(1/12)*tau^4-
(1/36)*tau^5+(1/16)*tau^6-(1/108)*tau^7-(1/864)*tau^8)*sin((j)*h)+...
            (-tau*(1/2)*exp(-1))*sin((j)*h);
   end;
    end;fii ;
```

```
    alpha(1:N+1,1:N+1,1:1) = 0*eye(N+1) ;
    betha(1:N+1,1:N+1,1:1) = 0*eye(N+1) ;
    gamma(N+1,1:1)= 0 ;
    alpha(1:N+1,1:N+1,2:2) = (4/5)*eye(N+1) ;
    betha(1:N+1,1:N+1,2:2) = (-1/5)*eye(N+1);
    gamma(N+1,2:2)= 0 ;


for n = 2:M-2 ;
bebek = C + D*alpha(:,: , n:n ) + E*betha(:,:,n-1 : n-1)+
E*alpha(:,:,n-1:n-1)*alpha(:,:, n:n) ;
betha(:,:,n+1:n+1 ) = -inv( bebek )*(A) ;
alpha(:,:,n+1:n+1) = -inv(bebek )*(B +D*betha(:,:,n:n)+ E *
alpha(:,:,n-1:n-1)* betha(:,:,n) ) ;
gamma(:,n+1:n+1) = inv( bebek )*(R*fii(:,n:n) - D * gamma(:,n:n)-E *
alpha(:,:,n-1:n-1)* gamma(:,n:n) - E*gamma(:, n-1 : n-1) ) ;
end;
U(1:N+1,1:N+1)=nan;
U( 1:N+1, M:M ) = 0 ;
U( :, M-1:M-1 ) = inv( (betha(:,:,M-2:M-2) + 5*eye(N+1))- (4*eye(N+1)-
alpha(:,:,M-2:M-2) )*alpha(:,:,M-1:M-1))*((4*eye(N+1)-alpha(:,:,M-2:M-
2))*gamma(: , M-1:M-1)- gamma(: , M-2:M-2) );
U(: , M-2:M-2 ) =inv(4*eye(N+1)-alpha(:,:,M-2:M-2))*((betha(:,:,M-2:M-
2)+5*eye(N+1))*U(:,M-1:M-1)+gamma(:,M-2:M-2));

for z = M-3:-1:1;
U(:,z:z
)=alpha(:,:,z+1:z+1)*U(:,z+1:z+1)+betha(:,:,z+1:z+1)*U(:,z+2:z+2)+gamm
a(:,z+1:z+1);end;

for z = 1 : M ;
p(:,z+1:z+1)=U(:,z:z);
end;

 'EXACT SOLUTION OF THIS PROBLEM' ;
    for d=1:N+1;
        for f=1:M+1 ;
    es( d, f )=exp(-(d-1)*tau)* sin((f-1)*h);
    end;
    end;es;


    figure ;
n(1,1)=min(min(es))-0.01;
n(2,2)=nan;
surf(n);
hold;
surf(es) ; rotate3d ;axis tight;
title('The exact solution');
figure ;
m(1,1)=min(min(p))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(p) ; rotate3d ;axis tight;
title ('The fourth order of accuracy difference scheme solution');
```

```
% .ERROR ANALYSIS.;
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)));
relativeerror=max(max((abs(es-p))))/max(max(abs(p)) );
cevap = [maxerror,relativeerror]
```

ÖZGEÇMİŞ

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